Derivation of Euler's Formula by Integration

Start with: $y = \cos x + i \sin x$

Then: $dy = (-\sin x + i \cos x)\ dx$

$dy = (i \cos x - \sin x)\ dx$

$dy = iy\ dx$

$\frac{dy}{y} = i\ dx$

Integrate: $\int \frac{dy}{y} = \int i\ dx$

$\ln y = i\,x$

$y = e^{ix}$

Final Result:

$$e^{ix} = \cos x + i \sin x$$

Very cool sub-case

When $\boldsymbol{x = \pi}$, Euler's equation becomes:

$e^{i\pi} = \cos \pi + i \sin \pi$

or, $e^{i\pi} = -1$

Rewriting this provides an equation that relates 5 of the most important mathematical constants to each other:

$$e^{i\pi} + 1 = 0$$

Derivation of Euler's Formula Using Power Series

A Power Series about zero is an infinite series of the form:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Many mathematical functions can be expressed as power series. Of particular interest in deriving Euler's Identity are the following:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

Note, then, that:

$$\mathrm{i} \cdot \sin(x) = i \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = ix - \frac{i \cdot x^3}{3!} + \frac{i \cdot x^5}{5!} - \frac{i \cdot x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix - \frac{x^2}{2!} - \frac{i \cdot x^3}{3!} + \frac{x^4}{4!} + \frac{i \cdot x^5}{5!} - \frac{x^6}{6!} - \frac{i \cdot x^7}{7!} + \cdots$$

Notice that the first two power series add to the third, so we have:

$$e^{ix} = \cos x + i \sin x$$

and, substituting $x = \pi$ yields: →

$$e^{i\pi} + 1 = 0$$