HIGHEST COMMON FACTORS USING LINEAR COMBINATIONS

We will be using **natural numbers N** in the following discussion. **M, N, a, b, c, x and y** will represent elements of **N={1,2,3,...}.**

Consider a **linear combination** of M and N, say **aM+bN.** Let **c** be a **common factor** of M and N. Then there exist **x** and **y** such that **M=cx** and **N=cy.** This makes **aM+bN = acx+bcy = c(ax+by)** so that **c** is seen to be a factor of aM+bN.

This fact can be used to provide a **really nice way** to find the **HCF** (HCF = GCD, the Greatest Common Divisor) of two numbers M and N **without** having to do a prime factorization of M and N.

Example 1) Given **M=48** and **N=52.** The difference **N-M=4** is a linear combination of M and N and so must have the HCF of M and N as a factor. But the only factors of 4 are 4, 2 and 1. Since 4 divides both M and N, **HCF(M,N)=4.**

Example 2) Given **M=50** and **N=54. N-M=4** again so the possibilities for HCF(M,N) are 4, 2 and 1. 2 divides both M and N, but 4 does not. Hence **HCF(M,N)=2.**

Example 3) Given **M=53** and **N=49. M-N=4** yet again so 4, 2 and 1 are the possibilities for HCF(M,N). Since 2 does not divide both M and N then neither can 4. Hence **HCF(M,N)=1.**

Example 4) Given **M=48** and **N=100. N-2*M**=100-96 = 4 is a linear combination of M and N. So the possibilities for HCF(M,N) are again 4, 2 and 1. **4=HCF(M,N)** since 4 divides both M and N.

Example 5) **M=203** and **N=303. 3*M-2*N**=3*203-2*303=609-606=3 so the only possibilities for HCF(M,N) are 3 and 1. Since 3 does not divide 203, HCF(M,N)=1.

Example 6) **M=30** and **N=42. 3*M-2*N**=90-84=6 so 6, 3, 2 and 1 are the only possibilities for HCF(M,N). Both divide by 2 and 3 so both divide by 6. Hence **HCF(M,N)=6.**

Example 7) **M=63** and **N=51. M-N=12** so the possibilities are 12, 6, 4, 3, 2 and 1. Since at least one of M and N does not divide by 2 then 12, 6, 4 and 2 are **not** possibilities. This leaves only 3 and 1. The sum of the digits of both M and N divide by 3. Hence M and N both divide by 3. **HCF(M,N)=3.**

Example 8) **M=36** and **N=64.** The SUM **M+N**=100 is a linear combination of M and N so the factors of 100 are the **only** possibilities for HCF(M,N). So 100, 50, 25, 20, 10, 5, 4, 2 and 1 are the choices for HCF(M,N). 5 and all the multiples of 5 are out since 5 does not divide both M and N. Hence 4, 2 and 1 are all the choices left. 4 divides both M and N so **HCF(M,N)=4**.

Example 9) **M=100** and **N=28.** The sum **M+N**=128 is a power of 2, so the only possibilities are 1, 2, 4, 8, 16. (32, 64 and 128 are definitely not possible since there are greater than 28.) 4 (but not 8) divides both M and N, so **HCF(M,N)=4**.

We can **use this technique for 3 (or more) numbers** as well. Just find the HCF of two of the numbers and then check the factors of this HCF against the third number. To **decrease** the number of possibilities, choose two of the numbers in the set that produce the **smallest** linear combination.

Example 10) **M=24**, **N=30**, **L=32**. L-N=2 so 2 and 1 are the choices for the HCF of all three. 2 divides all three so **HCF(M,N,L)=2**.

Example 11) **M=48, N=36, L=95**. 2*M-L=96-95=1 so **HCF(M,N,L)=1**.

Example 12) **M=40, N=60, L=97**. 97 is prime and also the largest of the three numbers, so **HCF(M,N,L)=1.**

Recall that given **just two** numbers M and N the product of M and N equals the product of HCF(M,N) and LCM(M,N). Hence we can find the Least Common Multiple of M and N by dividing the product by the HCF; that is, **LCM(M,N)=M*N/HCF(M,N)**.

Example: 13) **M=30** and **N=48**. HCF(M,N) is a factor of N-M=12. So the choices are 12, 6, 4, 3, 2 and 1. 2 and 3 both divide M and N but 30 does not divide by 12, so **HCF(M,N)=6**. Then we can calculate the LCM: **LCM = (30*48)/6 = 240**.

Given a set of more than two numbers to find the HCF (or LCM) we can select ***two*** *of the numbers and find HCF. Then take this result and another number in the set and find the HCF.* Doing this until there are no more numbers in the set will produce the HCF for the whole set. A similar approach can be used for the LCM of the set of numbers.

Of course there are cases where we may **prefer** to use other methods for finding HCF and LCM, but this provides a nice alternative in **many** (perhaps *most*) **cases.**