# Algebra
# Domains of Rational Functions

**Rational Expressions** are fractions with polynomials in both the numerator and denominator. If the rational expression is a function, it is a **Rational Function**.

## Finding the Domain of a Rational Function

The domain (e.g., $x$-values) of a rational function is the set of all values that result in valid range values (e.g., $y$-values). Generally, there are two situations where a value is not included in the domain of a rational function:

- Any $x$ that generates a zero in the denominator.
- Any $x$ that generates a square root of a negative number.

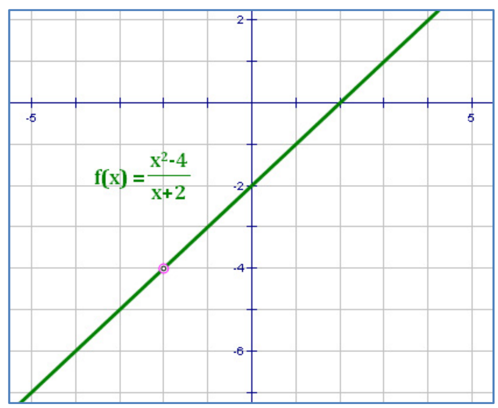
**Example 1:**

Consider the rational function: $\boldsymbol{f(x) = \frac{x^2-4}{x+2}}$.

Since there are no square roots, the only value for which we cannot calculate $\boldsymbol{f(x)}$ is where $\boldsymbol{x+2=0}$ or, where $\boldsymbol{x=-2}$. So the domain is all real $\boldsymbol{x}$ except $\boldsymbol{x=-2}$, or:

$$\{\boldsymbol{x} \mid \boldsymbol{x \neq -2}\}$$

Notice the hole in the graph of the function at the point $(-2,-4)$. This indicates that the function does not have a value for $\boldsymbol{x=-2}$.

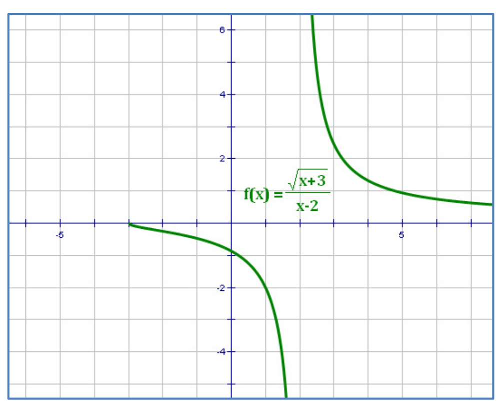



**Example 2:**

Consider the function: $\boldsymbol{f(x) = \frac{\sqrt{x+3}}{x-2}}$

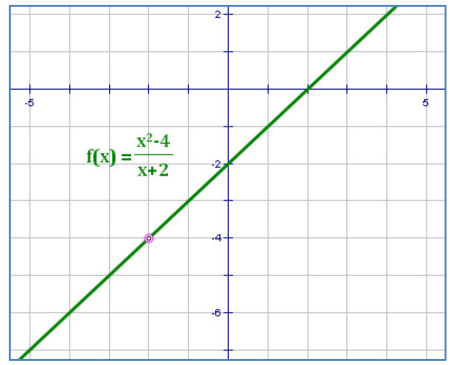
This function has no valid $x$-values for $x < -3$ because they would generate the square root of a negative number in the numerator. In addition, the denominator would be zero if $\boldsymbol{x=2}$. So the domain is all real $\boldsymbol{x}$ greater than **-3** except $\boldsymbol{x=2}$, or:

$$\{\boldsymbol{x} \mid \boldsymbol{x} > -3 \quad and \quad x \neq \boldsymbol{2}\}$$



# Algebra
# Holes and Asymptotes

## Holes

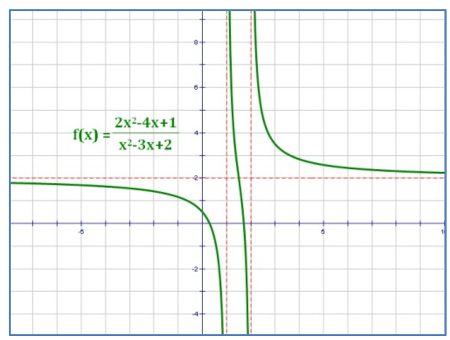



A **hole in a graph** exists whenever a factor $(x - a)$ occurs more times in the numerator than in the denominator of a rational function.

**Example:** In the function $f(x) = \frac{x^2-4}{x+2}$ the factor $(x+2)$ is in both the numerator and the denominator. In fact, the function can be reduced to $f(x) = x - 2$ except at the point $x = -2$ where the function is undefined.

## Vertical Asymptotes

A **vertical asymptote** exists whenever a factor $(x - a)$ occurs more times in the denominator than in the numerator of a rational function.




**Example:** In $f(x) = \frac{2x^2-4x+1}{x^2-3x+2} = \frac{2x^2-4x+1}{(x-1)(x-2)}$ the factors $(x-1)$ and $(x-2)$ occur in the denominator but not in the numerator of the function, so they generate vertical asymptotes. The vertical asymptotes are shown as red dotted lines at $x = 1$ and $x = 2$ in the graph at right.

## Horizontal Asymptotes

There are three separate cases for **horizontal asymptotes** of a rational function $\frac{P(x)}{Q(x)}$:

1. If **the degree of $P(x)$ > the degree of $Q(x)$**, there is no horizontal asymptote.
2. If **the degree of $P(x)$ = the degree of $Q(x)$,** a horizontal asymptote exists at the line:

   $$y = \frac{\textit{lead coefficient of } P(x)}{\textit{lead coefficient of } Q(x)}.$$

3. If **the degree of $P(x)$ < the degree of $Q(x)$**, a horizontal asymptote exists at the line $y = 0$.

**Example:** In the function $f(x) = \frac{2x^2-4x+1}{x^2-3x+2}$ the degrees of the polynomials in the numerator and denominator are the same, and the ratio of their lead coefficients is $\frac{2}{1} = 2$. The location of the horizontal asymptote is shown as the red dotted line $y = 2$ in the graph above.

# Algebra
## Graphing Rational Functions

**Rational functions** are of two types:

- **Simple rational functions** are of the form $\boldsymbol{y = \frac{a}{x-h} + k}$ or an equivalent form that does not contain a polynomial of degree higher than 1 (i.e., no $x^2, x^3$, etc. – just $x$'s and constants).

- **General rational functions** are of the form $\boldsymbol{y = \frac{P(x)}{Q(x)}}$ where either $P(x)$ or $Q(x)$ is a polynomial of degree 2 or higher (i.e., contains an $x^2, x^3, etc.$).

In general, it is a good idea to find the asymptotes for a function first, and then find points that help graph the curve. The domain and any holes can typically be easily identified during this process. The range and the end behavior become identifiable once the function is graphed.

## Simple Rational Functions

If you can put a rational function in the form $\boldsymbol{y = \frac{a}{x-h} + k}$, here's what you get:

**Vertical Asymptote:** Occurs at $\boldsymbol{x = h}$. The vertical asymptote is easy to find because it occurs at $x = h$. At this value of $x$, the denominator is $h - h = 0$, and you cannot divide by zero. Hence, as $x$ approaches $h$, the denominator of $\frac{a}{x-h}$ becomes very small, and the graph shoots off either up or down.

**Horizontal Asymptote:** Occurs at $\boldsymbol{y = k}$. The function cannot have a value of $y = k$ because that would require the lead term, $\frac{a}{x-h}$ to be zero, which can never happen since $a \neq 0$. Hence, the function will approach $y = k$, but will never reach it.

**Domain:** All Real $\boldsymbol{x \neq h}$. No value of $x$ exists at any vertical asymptote.

**Range:** All Real $\boldsymbol{y \neq k}$. No value of $y$ exists at a horizontal asymptote in simple rational functions.

**Holes: None.**

**End Behavior:** Both ends of the function tend toward the horizontal asymptote, so:

$\boldsymbol{As\ x \to -\infty, y \to k}$ and $\boldsymbol{As\ x \to \infty, y \to k}$

# Simple Rational Functions - Example

**Example:** $y = \frac{5}{x-1} - 2$

***First, note that*** $h = 1$ ***and*** $k = -2$

> Recall that the simple rational form is: $y = \frac{a}{x-h} + k$

**Vertical Asymptote:** Occurs at $x = 1$ because if $x = 1$, the denominator, $x - 1$, would be zero.

**Horizontal Asymptote:** Occurs at $y = -2$ because the lead term, $\frac{5}{x-1}$, can never be zero. Hence, the function can approach $y = -2$, but will never reach it.

**Domain:** All Real $x \neq 1$. No value of $x$ exists at any vertical asymptote.

**Range:** All Real $y \neq -2$. No value of $y$ exists at a horizontal asymptote in a simple rational function.

**Holes:** None.

**End Behavior:** Both ends of the function tend toward the horizontal asymptote, so:

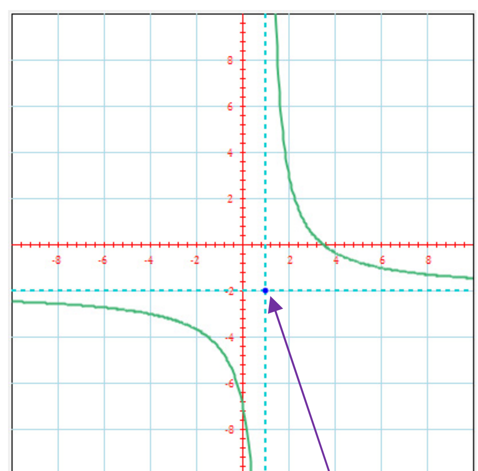
$As\ x \to -\infty, y \to -2$ and $As\ x \to \infty, y \to -2$

**Graphing:**

**Step 1.** Graph the vertical and horizontal asymptotes (the dashed horizontal and vertical lines shown).

**Step 2.** Pick some $x$-values and calculate the corresponding $y$-values. I like to pick a couple of $x$-values to the left of the vertical asymptote ($x = 1$) and a couple of $x$-values to its right. So, let's try some.

| $x$ | $y = \frac{5}{x-1} - 2$ |
|---|---|
| -2 | $-3.67$ |
| -1 | $-4.5$ |
| 0 | $-7$ |
| 2 | 3 |
| 3 | 0.5 |
| 4 | $-0.33$ |



*Note that the intersection of the asymptotes has coordinates* $(h, k)$.

**Step 3.** Draw a curve on each side of the vertical asymptote: through the points on that side and approaching both the horizontal and vertical asymptotes.

# Algebra
## General Rational Functions

**General rational functions** are of the form: $\boldsymbol{y = \frac{P(x)}{Q(x)}}$

The easiest way to graph a general rational function is to factor both the numerator and denominator and simplifying the resulting fraction.

**Example:** in $y = \frac{(x+3)(x+2)}{(x-1)(x+2)}$ the $(x+2)$ in the numerator and denominator can be eliminated to obtain the function to be graphed: $y = \frac{(x+3)}{(x-1)}$.

**Vertical Asymptotes and Holes:** Any root (also called a "zero") of the denominator of a rational function (prior to simplification) will produce either a vertical asymptote or a hole.

**Vertical Asymptote:** If $r$ is a root of the denominator is also a root of the <u>simplified denominator</u>, then $x = r$ is a vertical asymptote of the function.

**Hole:** If $r$ is a root of the denominator and is not a root of the <u>simplified denominator</u>, then $x = r$ defines the location of a hole in the function.

**Horizontal Asymptote:** One way to find the horizontal asymptotes of a general rational function (also, see the section on *"Holes and Asymptotes"*, above) is to eliminate all terms of the polynomials in both the numerator and denominator except the ones with the single greatest exponent of all the terms. Then,

- If all terms are eliminated from the numerator, the horizontal asymptote occurs at $y = 0$.

  **Example:** $\boldsymbol{y = \frac{x+3}{x^2-5x+6} \quad \rightarrow \quad y = \frac{nothing}{x^2}}$ has a horizontal asymptote at $y = 0$.

  Note that all terms in the numerator were eliminated because none of them had the greatest exponent in the rational function, which in this example is 2.

- If a term remains in both the numerator and denominator, the horizontal asymptote occurs at the reduced form of the remaining terms.

  **Example:** $\boldsymbol{y = \frac{2x^2+3}{3x^2-5x+6} \quad \rightarrow \quad y = \frac{2x^2}{3x^2} = \frac{2}{3}}$ has a horizontal asymptote at $y = \frac{2}{3}$.

- If all terms are eliminated from the denominator, the function does not have a horizontal asymptote.

  **Example:** $\boldsymbol{y = \frac{x^2-5x+6}{x-3} \quad \rightarrow \quad y = \frac{x^2}{nothing}}$ does not have a horizontal asymptote.

  Note that all terms in the denominator were eliminated because none of them had the greatest exponent in the rational function, which in this example is 2.

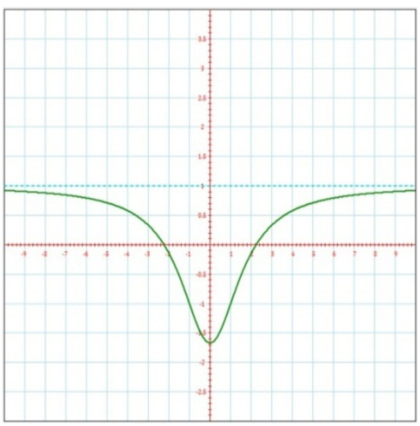
# Algebra
## General Rational Functions (cont'd)

**Domain:** The domain is always "all Real $x$" except where there is a vertical asymptote or a hole. No function value is associated with $x$ at either a vertical asymptote or a hole (or when an even root of a negative number is required).

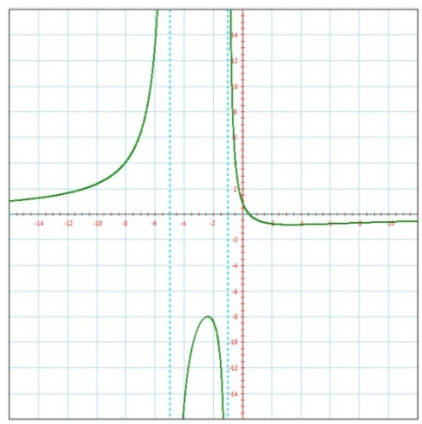
**Range:** The range is a bit trickier. You will need to look at the graph to determine the range. You might think that no $y$-value would exist at a horizontal asymptote, like in simple rational functions. However, it is possible for a function to cross over its horizontal asymptote and then work its way back to the asymptote as $x \to -\infty$ or as $x \to \infty$. Odd but true (see below, right).

For oddities in the range of a function, check these out these two rational functions:

$$y = \frac{x^2-5}{x^2+3}$$

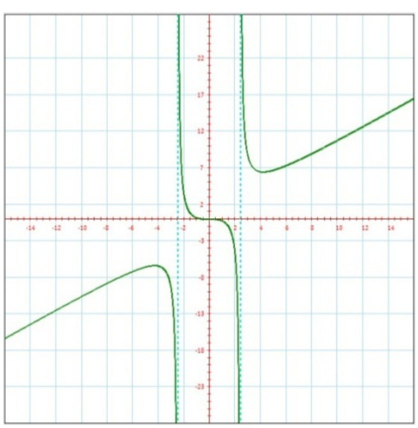


$$y = \frac{-10x+5}{x^2+6x+5}$$



**End Behavior:** Both ends of the function tend toward the horizontal asymptote if there is one. However, if there is not one, you can look at the graph to determine end behavior. Note that the function below does not have a horizontal asymptote:

$$y = \frac{x^3}{x^2-6}$$



In this function,

$\text{As } x \to -\infty, y \to -\infty,$

$\text{As } x \to \infty, y \to \infty$

Although this function does not have a horizontal asymptote, it does have a "slant asymptote": the line $y = x$.

# Algebra
## General Rational Functions - Example

**Example:** $y = \frac{2x^2+x-3}{x^2-1}$

**Factor both the numerator and the denominator:** $y = \frac{(x-1)(2x+3)}{(x+1)(x-1)}$

**Get the Roots:** $\frac{Numerator:\ x=-1.5,\ \ x=1}{Denominator:\ x=-1,\ \ x=1}$

**Simplify:** Since $1$ is a root of both the numerator and the denominator, the function may be simplified as follows:

$$y = \frac{(x-1)(2x+3)}{(x+1)(x-1)} = \frac{\cancel{(x-1)}(2x+3)}{(x+1)\cancel{(x-1)}} = \frac{2x+3}{x+1}$$

**Vertical Asymptotes and Holes:** "$-1$" and "$1$" are roots of the original denominator, so they must generate either vertical asymptotes or holes.

> **Vertical Asymptote:** After simplification, this function still contains "$-1$" as a root in the denominator. Therefore, $\boldsymbol{x = -1}$ **is a vertical asymptote** of the function.
>
> **Hole:** "$1$" is a root of the denominator of the original function but is not a root of the denominator of the simplified function. Therefore, this function has **a hole at $\boldsymbol{x = 1}$.**

**Horizontal Asymptote:** Eliminate all terms of both polynomials except any with the single greatest exponent of all the terms. In this case:

> $\boldsymbol{y = \frac{2x^2}{x^2}} \quad \rightarrow \quad \boldsymbol{y = 2}$ **is a horizontal asymptote.** Since a term remains in both the numerator and denominator, the horizontal asymptote occurs at the reduced form of the remaining terms.

**Domain:** All Real $x$ except where there is a vertical asymptote or a hole.
So, the domain is **all Real $\boldsymbol{x \neq -1\ or\ 1}$.**

We must graph the function in order to get a good look at its **range and end behavior**. We must plot points on both sides of the vertical asymptote.

*(graph on next page)*

### Graphing:

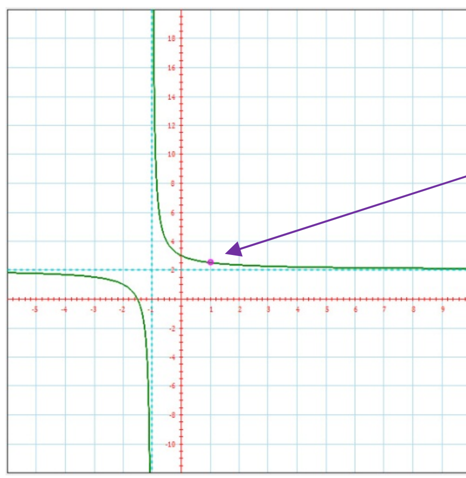
**Step 1.** Graph the vertical and horizontal asymptotes.

**Step 2.** Pick some $x$-values on each side of the vertical asymptote and calculate the corresponding $y$-values.

| $x$ | $y = \frac{2x+3}{x+1}$ |
|---|---|
| -4 | 1.67 |
| -3 | 1.5 |
| -2 | 1 |
| 0 | 3 |
| 1 | 2.5 (a hole) |
| 2 | 2.33 |

**Step 3.** Draw a curve on each side of the vertical asymptote: through the points on that side and approaching both the horizontal and vertical asymptotes.

**Step 4:** Draw an open circle at the point of any holes.



***See the hole at $(1, 2.5)$!***

**Range:** The range can be determined from the graph.
It appears that the range excludes only the horizontal asymptote and the hole.
So the range is: **all Real $y \neq 1, 2.5.$**

**End Behavior:** In this function,

$$As\ x \to -\infty, y \to 1, \qquad As\ x \to \infty, y \to 1$$