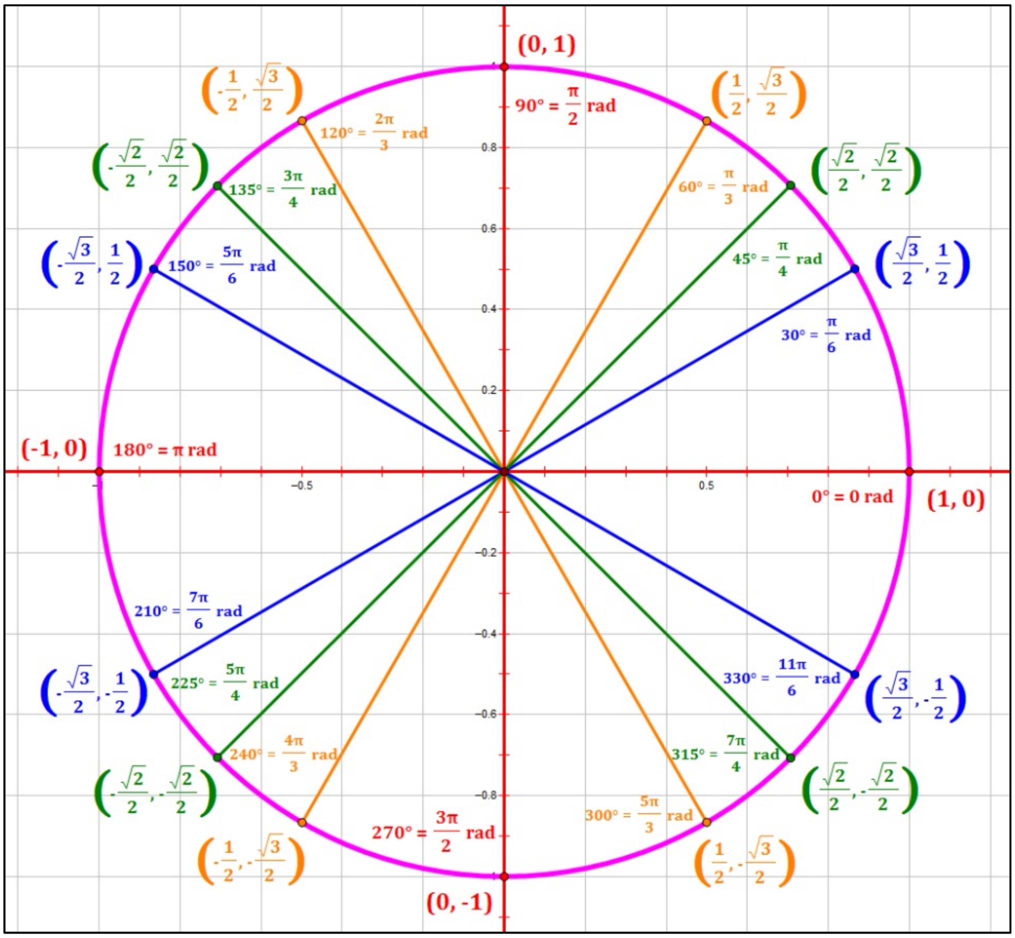
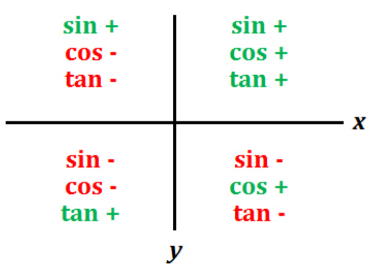
Trigonometry (Chapter 6) – Sample Test #1

First, a couple of things to help out:



Trig Functions of Special Angles (θ)				
Radians	**Degrees**	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0°	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{\sqrt{4}} = 0$
$\pi/6$	30°	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\pi/4$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{\sqrt{2}} = 1$
$\pi/3$	60°	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{\sqrt{1}} = \sqrt{3}$
$\pi/2$	90°	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{2} = 0$	*undefined*

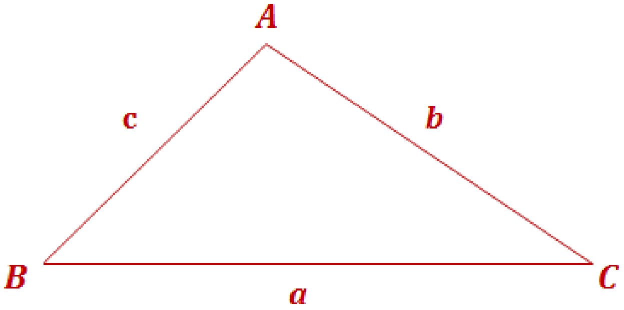
Signs of Trig Functions by Quadrant



More Formulas (memorize these):

Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\ \cos A$$

$$b^2 = a^2 + c^2 - 2ac\ \cos B$$

$$c^2 = a^2 + b^2 - 2ab\ \cos C$$

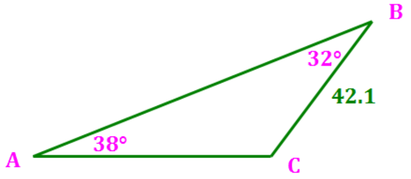
Area of a Triangle:

$$A = \frac{1}{2}ab\ \sin C = \frac{1}{2}ac\ \sin B = \frac{1}{2}bc\ \sin A$$

$$A = \sqrt{s(s-a)(s-b)(s-c)} \qquad s = \frac{1}{2}(a+b+c)$$

Solve the triangle. Round lengths to the nearest tenth and angle measures to the nearest degree.

1) A = 38°
B = 32°
a = 42.1



To solve: find the third angle, and then use the Law of Sines:

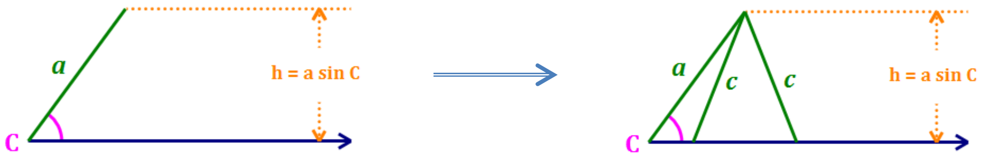
$$\boldsymbol{m\angle C} = 180° - 38° - 32° = \mathbf{110°}$$

$$\frac{42.1}{\sin 38°} = \frac{b}{\sin 32°} \quad \Rightarrow \quad \boldsymbol{b} = \frac{42.1 \cdot \sin 32°}{\sin 38°} = \mathbf{36.2}$$

$$\frac{42.1}{\sin 38°} = \frac{c}{\sin 110°} \quad \Rightarrow \quad \boldsymbol{c} = \frac{42.1 \cdot \sin 110°}{\sin 38°} = \mathbf{64.3}$$

Two sides and an angle (SSA) of a triangle are given. Determine whether the given measurements produce one triangle, two triangles, or no triangle at all. Solve each triangle that results. Round lengths to the nearest tenth and angle measures to the nearest degree.

2) C = 35°
a = 18.7
c = 16.1



First, calculate the measure of $\angle A$ using the Law of Sines:

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \Rightarrow \quad \frac{18.7}{\sin A} = \frac{16.1}{\sin 35°} \quad \Rightarrow \quad \sin A = 0.6662$$

So, $m\angle A = \sin^{-1} 0.6662 = 42°$ or $m\angle A = 180° - 42° = 138°$

Note that $(c = 16.1) < (a = 18.7)$, so we must compare c to the height of the triangle:

$$h = a \sin C = 18.7 \sin 35° = 10.7 \quad \text{so that} \quad (h = 10.7) < (c = 16.1)$$

Then, $h < c < a$, so we will have two triangles and must solve each.

Triangle 1 – Start with:

$a = 18.7,\ \ c = 16.1$
$m\angle C = 35°,\ \ \boldsymbol{m\angle A = 42°}$

Then,

$\boldsymbol{m\angle B} = 180° - 35° - 42° = \boldsymbol{103°}$

And,

$$\frac{b}{\sin B} = \frac{c}{\sin C} \quad \Rightarrow \quad \frac{b}{\sin 103°} = \frac{16.1}{\sin 35°}$$

$\boldsymbol{b = 27.4}$

Triangle 2 – Start with:

$a = 18.7,\ \ c = 16.1$
$m\angle C = 35°,\ \ \boldsymbol{m\angle A = 138°}$

Then,

$\boldsymbol{m\angle B} = 180° - 35° - 138° = \boldsymbol{7°}$

And,

$$\frac{b}{\sin B} = \frac{c}{\sin C} \quad \Rightarrow \quad \frac{b}{\sin 7°} = \frac{16.1}{\sin 35°}$$

$\boldsymbol{b = 3.4}$

3) B = 88°
b = 2
a = 23

First, calculate the value of A using the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \Rightarrow \quad \frac{23}{\sin A} = \frac{2}{\sin 88°} \quad \Rightarrow \quad \sin A = 11.5$$

11.5 is not in the range of the function $f(x) = \sin x$. Therefore, **the given values do not define a triangle.**

Find the area of the triangle having the given measurements. Round to the nearest square unit.

4) A = 37°, b = 10 inches, c = 9 inches

$$\boldsymbol{Area} = \frac{1}{2}\, bc\, \sin A = \frac{1}{2} \cdot 10 \cdot 9 \cdot \sin 37° = \textbf{27 inches}^{2}$$

Use Heron's formula to find the area of the triangle. Round to the nearest square unit.

5) a = 10 yards, b = 11 yards, c = 15 yards

$$Area = \sqrt{s(s-a)(s-b)(s-c)} \qquad s = \frac{1}{2}(a+b+c)$$

s is called the semi-perimeter because it is half of the perimeter.

First calculate: $s = \frac{1}{2}(10 + 11 + 15) = 18$

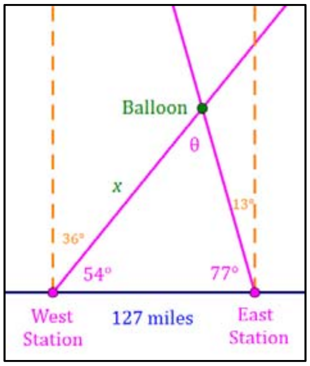
Then, $Area = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{18(18-10)(18-11)(18-15)}$

$$= \sqrt{18 \cdot 8 \cdot 7 \cdot 3} = \mathbf{12\sqrt{21}} \sim \textbf{ 55 yards}^{2}$$

Solve the problem.

6) Two tracking stations are on the equator 127 miles apart. A weather balloon is located on a bearing of N36°E from the western station and on a bearing of N13°W from the eastern station. How far is the balloon from the western station? Round to the nearest mile.

The angles given are those shown in orange in the diagram at right. The first step is to calculate the angles shown in magenta in the diagram.



$$90° - 36° = 54°$$

$$90° - 13° = 77°$$

$$\theta = 180° - 54° - 77° = 49°$$

Then, use the Law of Sines, as follows:

$$\frac{127}{\sin 49°} = \frac{x}{\sin 77°} \qquad \Rightarrow \qquad \boldsymbol{x = 164 \text{ miles}}$$

Solve the triangle. Round lengths to the nearest tenth and angle measures to the nearest degree.

When you are given the lengths of two sides and the measure of the angle between them:

- Use the Law of Cosines to determine the third side length,
- Use the Law of Sines to determine the measure of one of the two unknown angles, and
- Subtract the two known angle measures from 180° to get the measure of the last unknown angle.

$$\left.\begin{aligned} a^2 &= b^2 + c^2 - 2bc\ \cos A \\ b^2 &= a^2 + c^2 - 2ac\ \cos B \\ c^2 &= a^2 + b^2 - 2ab\ \cos C \end{aligned}\right\} \text{Law of Cosines}$$

Note: in problems with a lot of calculations, it is a good idea to keep more accuracy than you are required to provide in the final answer, and round your answer at the end. This avoids the compounding of rounding errors throughout your calculations.

7) a = 6, c = 11, B = 109°

$$\boldsymbol{b} = \sqrt{6^2 + 11^2 - 2(6)(11)(\cos 109°)} \sim 14.14125 \sim \mathbf{14.1}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \Rightarrow \quad \frac{6}{\sin A} = \frac{14.14125}{\sin 109°} \quad \Rightarrow \quad \sin A = 0.4012 \quad \Rightarrow \quad \boldsymbol{m\angle A = 24°}$$

$$\boldsymbol{m\angle C} = 180° - 109° - 24° = \mathbf{47°}$$

8) a = 7, b = 7, c = 5

$$a^2 = b^2 + c^2 - 2bc\ \cos A$$

$$7^2 = 7^2 + 5^2 - 2(7)(5)(\cos A)$$

Simplifying, we get $49 = 84 - 70\cos A$

Then, we solve to get: $\cos A = \frac{5}{14}$, and so $\boldsymbol{m\angle A} = \mathbf{69°}$

Since this is an isosceles triangle (note that $a = b$), $\boldsymbol{m\angle B} = \mathbf{69°}$

$$\boldsymbol{m\angle C} = 180° - 69° - 69° = \mathbf{42°}$$

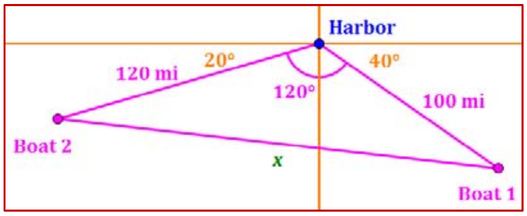
Solve the problem.

9) Two sailboats leave a harbor in the Bahamas at the same time. The first sails at 25 mph in a direction 320°. The second sails at 30 mph in a direction 200°. Assuming that both boats maintain speed and heading, after 4 hours, how far apart are the boats?

Boat 1 travels: $25 \text{ mph } \cdot 4 \text{ hours} = 100 \text{ mi.}$

Boat 2 travels: $30 \text{ mph } \cdot 4 \text{ hours} = 120 \text{ mi.}$

Using the Law of Cosines and the diagram at right, we can calculate:



$$x^2 = 100^2 + 120^2 - 2(100)(120)(\cos 120°) = 36{,}400 \quad \Rightarrow \quad \boldsymbol{x = 190.8 \text{ miles}}$$

Find another representation, (r, θ), for the point under the given conditions.

10) $\left(5, \frac{\pi}{6}\right)$, r < 0 and $2\pi < \theta < 4\pi$

$\left(5, \frac{\pi}{6}\right)$ can also be represented as $\left(-5, \frac{\pi}{6} + \pi\right) = \left(-5, \frac{7\pi}{6}\right)$

The resulting point is still in the interval $(0, 2\pi)$. To get to the interval $(2\pi, 4\pi)$, add 2π.

$$\left(-5, \frac{7\pi}{6} + 2\pi\right) = \boldsymbol{\left(-5, \frac{19\pi}{6}\right)}$$

Note: given the parameters of this question, another solution would be:

$$\left(5, \frac{\pi}{6} + 2\pi\right) = \left(5, \frac{13\pi}{6}\right)$$

Polar coordinates of a point are given. Find the rectangular coordinates of the point. Give exact answer.

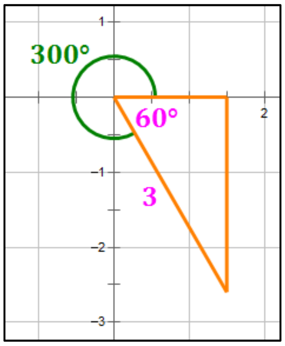
11) (-3, 120°)

First, note that $(-3, 120°) = (3, 120° + 180°) = (\overset{r}{3}, \overset{\theta}{300°})$

Then, $x = r\cos\theta = 3\cos 300° = 3 \cdot \frac{1}{2} = \frac{3}{2}$

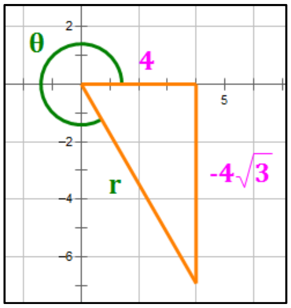
And, $y = r\sin\theta = 3\sin 300° = 3 \cdot \left(-\frac{\sqrt{3}}{2}\right) = \frac{-3\sqrt{3}}{2}$

So, the rectangular coordinates are: $\boldsymbol{\left(\frac{3}{2}, \frac{-3\sqrt{3}}{2}\right)}$



The rectangular coordinates of a point are given. Find polar coordinates of the point. Express θ in radians.

12) (4, -4$\sqrt{3}$)



$$r = \sqrt{4^2 + \left(4\sqrt{3}\right)^2} = 8$$

$$\theta = \tan^{-1}\left(\frac{-4\sqrt{3}}{4}\right) = \tan^{-1}(-\sqrt{3}) \text{ in } Q4 = \frac{5\pi}{3}$$

So, the polar coordinates are: $\left(\mathbf{8}, \frac{\mathbf{5\pi}}{\mathbf{3}}\right)$

Convert the rectangular equation to a polar equation that expresses r in terms of θ.

13) $x = 4$

Since $x = r\cos\theta$, we make that substitution and solve for r.

$$r\cos\theta = 4$$

$$\boldsymbol{r = \frac{4}{\cos\theta}}$$

14) $x^2 + y^2 = 16$

Substitute $x = r\cos\theta$ and $y = r\sin\theta$.

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 16$$

$$r^2(\cos^2\theta + \sin^2\theta) = 16$$ (recall that $\cos^2\theta + \sin^2\theta = 1$)

$$r^2 = 16$$

$$\boldsymbol{r = 4}$$ note that we take the positive root of r only

15) $8x - 3y + 10 = 0$

Substitute $x = r\cos\theta$ and $y = r\sin\theta$.

$$8 \cdot r\cos\theta - 3 \cdot r\sin\theta + 10 = 0$$

$$r\,(8\cos\theta - 3\sin\theta) = -10$$

$$\boldsymbol{r = \frac{-10}{8\cos\theta - 3\sin\theta}}$$

Convert the polar equation to a rectangular equation.

16) r = 5

Substitute $r = \sqrt{x^2 + y^2}$

$\sqrt{x^2 + y^2} = 5$

$\boldsymbol{x^2 + y^2 = 25}$

17) r = -3 cos θ

Substitute $\cos\theta = \frac{x}{r}$ and $r^2 = x^2 + y^2$

$r = -3\left(\frac{x}{r}\right)$

$r^2 = -3x$

$x^2 + y^2 = -3x$

$x^2 + 3x \qquad + y^2 = 0$

$\left(x^2 + 3x + \frac{9}{4}\right) + y^2 = \frac{9}{4}$

$\boldsymbol{\left(x + \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}}$

18) r = 8 cos θ + 9 sin θ

Substitute $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$ and $r^2 = x^2 + y^2$

$r = 8\left(\frac{x}{r}\right) + 9\left(\frac{y}{r}\right)$

$r^2 = 8x + 9y$

$x^2 + y^2 = 8x + 9y$

$x^2 - 8x \qquad + y^2 - 9y \qquad = 0$

$(x^2 - 8x + 16) + \left(y^2 - 9y + \frac{81}{4}\right) = 16 + \frac{81}{4}$

$\boldsymbol{(x - 4)^2 + \left(y - \frac{9}{2}\right)^2 = \frac{145}{4}}$

Graph the polar equation.

19) $\mathrm{r} = 2 + 2\sin\theta$

This cardioid is also a limaçon of form $r = a + b\sin\theta$ with $a = b$. The use of the sine function indicates that the large loop will be symmetric about the y-axis. The $+$ sign indicates that the large loop will be above the x-axis. Let's create a table of values and graph the equation:

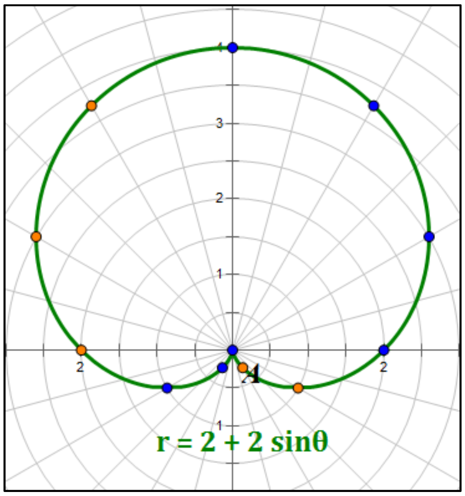
$\mathbf{r = 2 + 2\sin\theta}$			
$\boldsymbol{\theta}$	$\boldsymbol{r}$	$\boldsymbol{\theta}$	$\boldsymbol{r}$
0	2		
$\pi/6$	3	$7\pi/6$	1
$\pi/3$	3.732	$4\pi/3$	0.268
$\pi/2$	4	$3\pi/2$	0
$2\pi/3$	3.732	$5\pi/3$	0.268
$5\pi/6$	3	$11\pi/6$	1
π	2	2π	2

Generally, you want to look at values of θ in $[0, 2\pi]$. However, some functions require larger intervals. The size of the interval depends largely on the nature of the function and the coefficient of θ.

Once symmetry is established, these values are easily determined.

The portion of the graph above the x-axis results from θ in Q1 and Q2, where the sine function is positive.

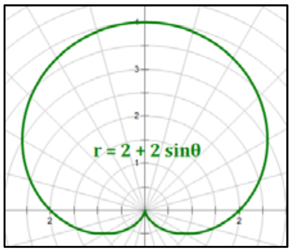
Similarly, the portion of the graph below the x-axis results from θ in Q3 and Q4, where the sine function is negative.

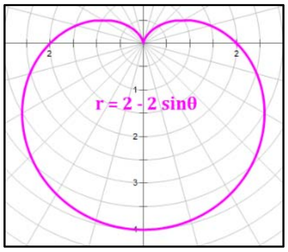


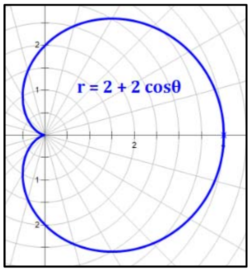
Blue points on the graph correspond to blue values in the table.

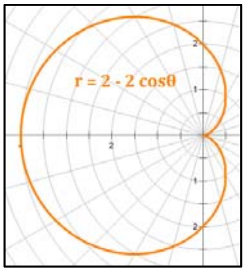
Orange points on the graph correspond to orange values in the table.

The four Cardioid forms:









20) $r = 4\sin 2\theta$

This function is a rose. Consider the forms $r = a\sin b\theta$ and $r = a\cos b\theta$. The number of petals on the rose depends on the value of b.

- If b is an even integer, the rose will have $2b$ petals.
- If b is an odd integer, it will have b petals.

The rose and other polar functions can be investigated in more detail using the Algebra App, available at www.mathguy.us.

Let's create a table of values and graph the equation:

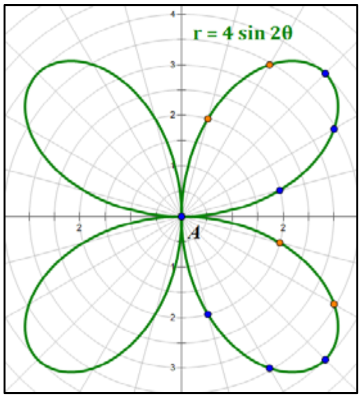
$\mathbf{r = 4\sin 2\theta}$			
$\boldsymbol{\theta}$	$\boldsymbol{r}$	$\boldsymbol{\theta}$	$\boldsymbol{r}$
0	0		
$\pi/12$	2	$7\pi/12$	-2
$\pi/6$	3.464	$2\pi/3$	-3.464
$\pi/4$	4	$3\pi/4$	-4
$\pi/3$	3.464	$5\pi/6$	-3.464
$5\pi/12$	2	$11\pi/12$	-2
$\pi/2$	0	π	0

Because this function involves an argument of 2θ**,** we want to start by looking at values of θ in $[0, 2\pi] \div 2 = [0, \pi]$. You could plot more points, but this interval is sufficient to establish the nature of the curve; so you can graph the rest easily.

Once symmetry is established, these values are easily determined.

The values in the table generate the points in the two petals right of the y-axis.

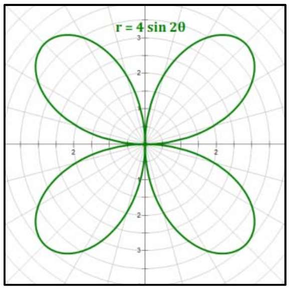
Knowing that the curve is a rose allows us to graph the other two petals without calculating more points.

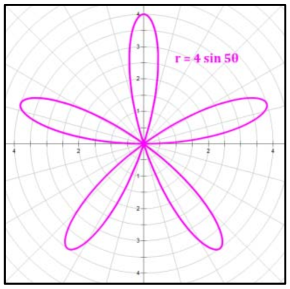


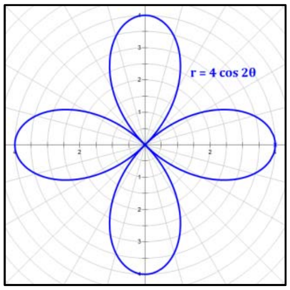
Blue points on the graph correspond to blue values in the table.

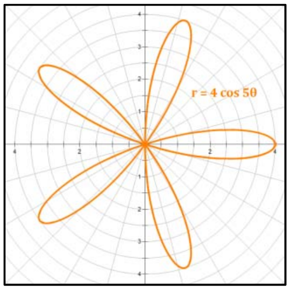
Orange points on the graph correspond to orange values in the table.

The four Rose forms:





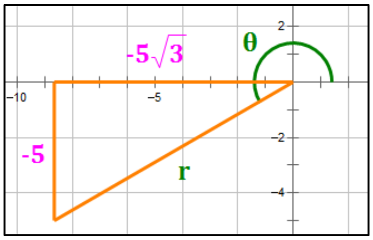




Write the complex number in polar form. Express the argument in radians.

21) $-5\sqrt{3}-5i$

The process of putting a complex number in polar form is very similar to converting a set of rectangular coordinates to polar coordinates. So, if this process seems familiar, that's because it is.



$$r=\sqrt{\left(-5\sqrt{3}\right)^2+(-5)^2}=10$$

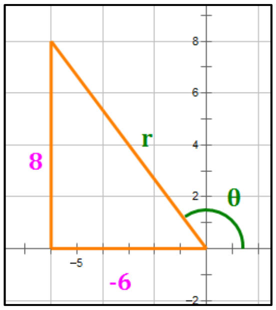
$$\theta=\tan^{-1}\left(\frac{-5}{-5\sqrt{3}}\right)=\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\text{ in }Q3=\frac{7\pi}{6}$$

So, the polar form is: $10\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)=\mathbf{10\ cis}\,\frac{\mathbf{7\pi}}{\mathbf{6}}$

Note: $10\text{ cis}\frac{7\pi}{6}=10e^{\left(i\cdot\frac{7\pi}{6}\right)}$ because $e^{i\theta}=\cos\theta+i\sin\theta$. However, the student may not be required to know this at this point in the course.

Write the complex number in polar form. Express the argument in degrees.

22) $-6+8i$



$$r=\sqrt{(-6)^2+(8)^2}=10$$

$$\theta=\tan^{-1}\left(\frac{8}{-6}\right)=\tan^{-1}\left(-\frac{4}{3}\right)\text{ in }Q2=126.9°$$

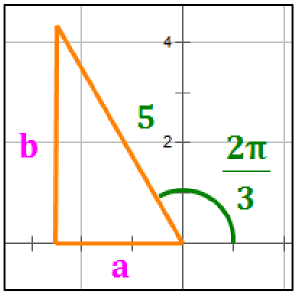
So, the polar form is: $\mathbf{10\ cis\ 126.9°}$

Note: To convert a polar form to exponential form, degrees must be converted to radians. So, $10\text{ cis }129.6°\sim10e^{(2.214i)}$. However, the student may not be required to know this at this point in the course.

Write the complex number in rectangular form. Give exact answer.

23) $5\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

We want this in the form $a+bi$



$$a=r\cos\theta=5\cos\frac{2\pi}{3}=5\cdot\left(-\frac{1}{2}\right)=-\frac{5}{2}$$

$$b=r\sin\theta=5\cos\frac{2\pi}{3}=5\cdot\left(\frac{\sqrt{3}}{2}\right)=\frac{5\sqrt{3}}{2}$$

So, the rectangular form is: $\mathbf{-\frac{5}{2}+i\,\frac{5\sqrt{3}}{2}}$

Find the product of the complex numbers. Leave answer in polar form.

24) $z_1 = 8\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$z_2 = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

To multiply two numbers in polar form, multiply the r-values and add the angles.

$$z_1 \cdot z_2 = 8 \cdot 3 \cdot \text{cis}\left(\frac{\pi}{6} + \frac{\pi}{2}\right) = \mathbf{24\ cis}\left(\frac{\mathbf{2\pi}}{\mathbf{3}}\right)$$

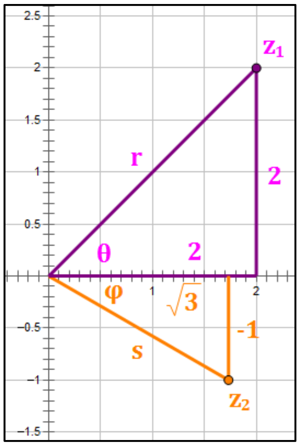
Note: this may be easier to understand in exponential form, since exponents are added when values with the same base are multiplied:

$$8e^{i\cdot\frac{\pi}{6}} \cdot 3e^{i\cdot\frac{\pi}{2}} = 24e^{i\left(\frac{\pi}{6}+\frac{\pi}{2}\right)} = 24e^{i\left(\frac{2\pi}{3}\right)}$$

25) $z_1 = 2 + 2i$

$z_2 = \sqrt{3} - i$

This problem is probably best approached by converting each number to polar form and then multiplying.



Relating to z_1:

$$r = \sqrt{(2)^2 + (2)^2} = 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) \text{ in } Q1 = \frac{\pi}{4}$$

So, the polar form is: $2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\sqrt{2}\ \mathbf{cis}\ \frac{\pi}{4}$

Relating to z_2:

$$s = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$$

$$\varphi = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) \text{ in } Q4 = \frac{-\pi}{6}$$

This angle could be $\frac{11\pi}{6}$ or $\frac{-\pi}{6}$. Thinking ahead, we know we will be adding it to $\frac{\pi}{4}$. By using $\frac{-\pi}{6}$, we will not need to simplify after adding.

So, the polar form is: $2\left(\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}\right) = 2\ \mathbf{cis}\ \frac{-\pi}{6}$

Then, multiply:

$$z_1 \cdot z_2 = \left(2\sqrt{2}\ \mathbf{cis}\ \frac{\pi}{4}\right) \cdot \left(2\ \mathbf{cis}\ \frac{-\pi}{6}\right) = 4\sqrt{2}\ \mathbf{cis}\left(\frac{\pi}{4} + \frac{-\pi}{6}\right) = \mathbf{4\sqrt{2}\ cis}\left(\frac{\boldsymbol{\pi}}{\mathbf{12}}\right)$$

Find the quotient $\frac{z_1}{z_2}$ of the complex numbers. Leave answer in polar form.

26) $z_1 = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$z_2 = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

To divide two numbers in polar form, divide the r-values and subtract the angles.

$$z_1 \cdot z_2 = \frac{8}{3}\cdot \text{cis}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \mathbf{\frac{8}{3}\ cis\left(\frac{\pi}{3}\right)}$$

Note: this may be easier to understand in exponential form, since exponents are subtracted when values with the same base are divided:

$$\frac{8e^{i\cdot\frac{\pi}{2}}}{3e^{i\cdot\frac{\pi}{6}}} = \frac{8}{3}e^{i\left(\frac{\pi}{2}-\frac{\pi}{6}\right)} = \frac{8}{3}e^{i\left(\frac{\pi}{3}\right)}$$

Use DeMoivre's Theorem to find the indicated power of the complex number. Write the answer in rectangular form.

27) $[3\,(\cos 15° + i\sin 15°)]^4$

To take a power of two numbers in polar form, take the power of the r-value and multiply the angle by the exponent. (This is the essence of DeMoivre's Theorem.)

$$[3\,(\cos 15° + i\sin 15°)]^4 = 3^4 \cdot \text{cis}(4\cdot 15°) = 81\ \text{cis}(60°)$$

$$81\ \text{cis}(60°) = 81\,(\cos 60° + i\sin 60°) = 81\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \mathbf{\frac{81}{2} + i\frac{81\sqrt{3}}{2}}$$

Note: Consider this in exponential form. Exponents are multiplied when a value with an exponent is taken to a power:

$$\left(3\ \text{cis}\ \frac{\pi}{12}\right)^4 = \left(3e^{i\cdot\frac{\pi}{12}}\right)^4 = 3^4e^{4\cdot\left(i\cdot\frac{\pi}{12}\right)} = 81\ e^{\left(i\cdot\frac{\pi}{3}\right)}$$

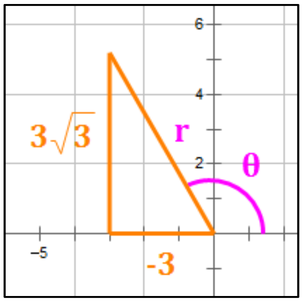
28) $(-3 + 3i\sqrt{3})^3$

$$r = \sqrt{(-3)^2 + \left(3\sqrt{3}\right)^2} = 6$$

$$\theta = \tan^{-1}\left(\frac{3\sqrt{3}}{-3}\right) = \tan^{-1}\left(-\sqrt{3}\right) \text{ in } Q2 = \frac{2\pi}{3}$$

Then, $(-3 + 3i\sqrt{3})^3 = \left(6\ \text{cis}\ \frac{2\pi}{3}\right)^3 = 6^3\ \text{cis}\left(3\cdot\frac{2\pi}{3}\right)$

$$= 216\ \text{cis}\ 2\pi = 216\cdot(\cos 2\pi + i\sin 2\pi) = \mathbf{216}$$



Find the specified vector or scalar.

29) **u** = -3**i** - 6**j**, **v** = 6**i** + 8**j**; Find **u** + **v**.

An alternative notation for a vector in the form $a\mathbf{i} + b\mathbf{j}$ is $\langle a, b\rangle$. Using this alternative notation makes many vector operations much easier to work with.

To add vectors, simply line them up vertically and add:

$$\begin{array}{r} \mathbf{u} = \langle -3, -6\rangle \\ \mathbf{v} = \langle \ \ 6, \ \ \ 8\rangle \\ \hline \mathbf{u} + \mathbf{v} = \langle -3 + 6, -6 + 8\rangle \end{array}$$

$$\mathbf{u} + \mathbf{v} = \langle \mathbf{3}, \mathbf{2}\rangle = \mathbf{3i} + \mathbf{2j}$$

30) **u** = -2**i** - 7**j** and **v** = -4**i** - 21**j**; Find $\|\mathbf{v} - \mathbf{u}\|$.

$$\begin{array}{r} \mathbf{v} = \langle -4, -21\rangle \\ + \ \ -\mathbf{u} = \langle \ \ 2, \ \ \ \ 7\rangle \\ \hline \mathbf{v} - \mathbf{u} = \langle \ \ 2, -14\rangle \end{array}$$

$$\|\mathbf{v} - \mathbf{u}\| = \sqrt{2^2 + (-14)^2}$$
$$= \sqrt{200} = \sqrt{100} \cdot \sqrt{2} = \mathbf{10\sqrt{2}}$$

Subtracting $\mathbf{u}$ is the same as adding $-\mathbf{u}$. To get $-\mathbf{u}$, simply change the sign of each element of $\mathbf{u}$. If you find it easier to add than to subtract, you may want to adopt this approach to subtracting vectors.

Find the unit vector that has the same direction as the vector v.

A unit vector has **magnitude 1**. To get a unit vector in the same direction as the original vector, divide the vector by its magnitude.

31) **v** = 5**i** - 12**j**

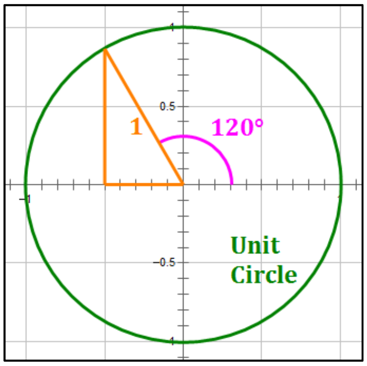
The unit vector is: $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{5\mathbf{i} - 12\mathbf{j}}{\sqrt{5^2 + 12^2}} = \dfrac{5\mathbf{i} - 12\mathbf{j}}{13} = \dfrac{\mathbf{5}}{\mathbf{13}}\mathbf{i} - \dfrac{\mathbf{12}}{\mathbf{13}}\mathbf{j}$

32) **v** = -8**j**

The unit vector is: $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{-8\mathbf{j}}{\sqrt{0^2 + (-8)^2}} = \dfrac{-8\mathbf{j}}{8} = -\mathbf{j}$

Write the vector v in terms of i and j whose magnitude v and direction angle θ are given. Give exact answer.

33) $\|\mathbf{v}\| = 10$, θ = 120°



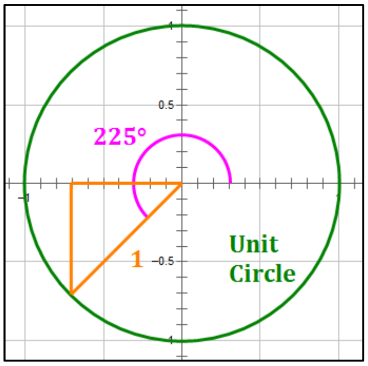
The unit vector in the direction $\theta = 120°$ is:

$$\langle \cos 120°, \sin 120° \rangle = \langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

Multiply this by $\|\mathbf{v}\| = \mathbf{10}$ to get $\mathbf{v}$:

$$\mathbf{v} = 10\left(-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right) = \mathbf{-5i + 5\sqrt{3}\,j}$$

34) $\|\mathbf{v}\| = 7$, θ = 225°



The unit vector in the direction $\theta = 225°$ is:

$$\langle \cos 225°, \sin 225° \rangle = \langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

Multiply this by $\|\mathbf{v}\| = \mathbf{7}$ to get $\mathbf{v}$:

$$\mathbf{v} = 7\left(-\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}\right) = -\frac{\mathbf{7\sqrt{2}}}{\mathbf{2}}\mathbf{i} - \frac{\mathbf{7\sqrt{2}}}{\mathbf{2}}\mathbf{j}$$

Solve the problem.

35) The magnitude and direction of two forces acting on an object are 65 pounds, N60°E, and 35 pounds, S45°E, respectively. Find the magnitude, to the nearest hundredth of a pound, and the direction angle, to the nearest tenth of a degree, of the resultant force.

This problem requires the addition of two vectors. The approach I prefer is:

1) Convert each vector into its $\mathbf{i}$ and $\mathbf{j}$ components, call them x and y,
2) Add the resulting x and y values for the two vectors, and
3) Convert the sum to its polar form.

Keep additional accuracy throughout and round at the end. This will prevent error compounding and will preserve the required accuracy of your final solutions.

Step 1: Convert each vector into its $\mathbf{i}$ and $\mathbf{j}$ components

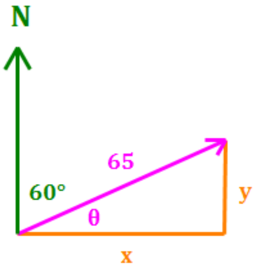
Let $\mathbf{u}$ be a force of 65 lbs. at bearing: $\mathrm{N}60°\mathrm{E}$

From the diagram at right,

$\theta = 90° - 60° = 30°$

$x = 65\cos 30° = 56.2917$

$y = 65\sin 30° = 32.5$



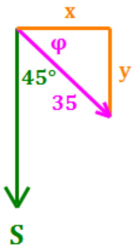
Let $\mathbf{v}$ be a force of 35 lbs. at bearing: $\mathrm{S}45°\mathrm{E}$

From the diagram at right,

$\varphi = -90° + 45° = -45°$

$x = 35\cos(-45°) = 24.7487$

$y = 35\sin(-45°) = -24.7487$



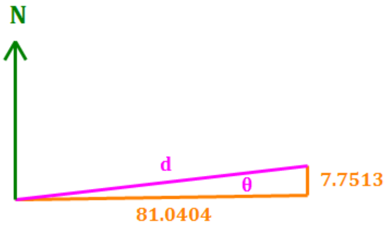
Step 2: Add the results for the two vectors

$$\begin{aligned} \mathbf{u} &= \langle 56.2917, \quad 32.5 \quad \rangle \\ \mathbf{v} &= \langle 24.7487, -24.7487 \rangle \\ \hline \mathbf{u}+\mathbf{v} &= \langle 81.0404, \quad 7.7513 \rangle \end{aligned}$$

Step 3: Convert the sum to its polar form

Direction Angle $= \theta = \tan^{-1}\left(\frac{7.7513}{81.0404}\right) = \mathbf{5.5°}$

Distance $= d = \sqrt{81.0404^2 + 7.7513^2} = \mathbf{81.41\ lbs.}$



Use the given vectors to find the specified scalar.

36) **u** = 6**i** + 5**j**, **v** = 6**i** - 9**j**,**w** = 7**i** - 3**j**; Find **u** · (**v** + **w**).

The alternate notation for vectors comes in especially handy in doing these types of problems:

$$\begin{array}{r} \mathbf{v} = \langle\ 6,\ -9\rangle \\ +\ \mathbf{w} = \langle\ 7,\ -3\rangle \\ \hline \mathbf{v}+\mathbf{w} = \langle 13, -12\rangle \\ \cdot\ \mathbf{u} = \langle\ 6,\quad 5\rangle \\ \hline \end{array}$$

$$\mathbf{u}\ \cdot\ (\mathbf{v}\ +\ \mathbf{w}) = (13\cdot 6)+(-12\cdot 5) = 78-60 = \mathbf{18}$$

Find the angle between the given vectors. Round to the nearest tenth of a degree.

37) **u** = -**i** + 6**j**, **v** = 4**i** - 4**j**

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\ \|\mathbf{v}\|}$$

$$0° \le \theta \le 180°$$

$$\begin{array}{r} \mathbf{u} = \langle -1,\ \ 6\rangle \\ \cdot\ \mathbf{v} = \langle\ \ 4, -4\rangle \\ \hline \end{array}$$

$$\mathbf{u}\ \cdot\ \mathbf{v} = (-1\cdot 4)+(6\cdot[-4]) = -28$$

$$\|\mathbf{u}\| = \sqrt{(-1)^2+6^2} = \sqrt{37}$$

$$\|\mathbf{v}\| = \sqrt{4^2+(-4)^2} = 4\sqrt{2}$$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\ \|\mathbf{v}\|} = \frac{-28}{\sqrt{37}\cdot 4\sqrt{2}} = \frac{-7}{\sqrt{74}}$$

$$\boldsymbol{\theta} = \cos^{-1}\left(\frac{-7}{\sqrt{74}}\right) = \mathbf{144.5°}$$

Use the dot product to determine whether the vectors are parallel, orthogonal, or neither.

If the vectors are parallel, one is a multiple of the other; also $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \ \|\mathbf{w}\|$.

If the vectors are perpendicular, their dot product is zero.

38) **v** = **i** + $\sqrt{3}$**j**, **w** = **i** - 4**j**

Neither vector is a multiple of the other.

$$\mathbf{v} = \langle 1, \sqrt{3} \rangle$$
$$\cdot\ \mathbf{w} = \langle 1, -4 \rangle$$
$$\mathbf{v} \cdot \mathbf{w} = (1 \cdot 1) + \left(\sqrt{3} \cdot [-4]\right) \neq 0$$

Therefore, **the vectors are neither parallel nor orthogonal**.

It is clearly easier to check whether one vector is a multiple of the other than to use the dot product method. The student may use either, unless instructed to use a particular method.

To determine if two vectors are parallel using the dot product, we check to see if:

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \ \|\mathbf{w}\|$$

$$\mathbf{v} = \langle 1, \sqrt{3} \rangle$$
$$\cdot\ \mathbf{w} = \langle 1, -4 \rangle$$
$$\mathbf{v} \cdot \mathbf{w} = (1 \cdot 1) + \left(\sqrt{3} \cdot [-4]\right) = 1 - 4\sqrt{3}$$

$$\|\mathbf{v}\| = \sqrt{(1)^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\|\mathbf{w}\| = \sqrt{(1)^2 + (-4)^2} = \sqrt{17}$$

$$\|\mathbf{v}\| \ \|\mathbf{w}\| = 2 \cdot \sqrt{17} = 2\sqrt{17} \neq \mathbf{v} \cdot \mathbf{w}$$

Therefore, the vectors are not parallel.

39) **v** = 3**i** - **j**, **w** = 6**i** - 2**j**

$$\mathbf{v} = \langle 3, -1 \rangle$$
$$\mathbf{w} = \langle 6, -2 \rangle$$

Clearly, $\mathbf{w} = 2\mathbf{v}$

Therefore, **the vectors are parallel.**

Using the dot product, we check to see if:

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \ \|\mathbf{w}\|$$

$$\mathbf{v} = \langle 3, -1 \rangle$$
$$\cdot\ \mathbf{w} = \langle 6, -2 \rangle$$
$$\mathbf{v} \cdot \mathbf{w} = 18 + 2 = 20$$

$$\|\mathbf{v}\| = \sqrt{(3)^2 + (-1)^2} = \sqrt{10}$$

$$\|\mathbf{w}\| = \sqrt{(6)^2 + (-2)^2} = \sqrt{40}$$

$$\|\mathbf{v}\| \ \|\mathbf{w}\| = \sqrt{10} \cdot \sqrt{40} = 20 = \mathbf{v} \cdot \mathbf{w}$$

Therefore, **the vectors are parallel.**

Decompose v into two vectors $\mathbf{v_1}$ and $\mathbf{v_2}$, where $\mathbf{v_1}$ is parallel to w and $\mathbf{v_2}$ is orthogonal to w.

40) **v** = **i** - 4**j**,**w** = 2**i** + **j**

The formulas for this are:

$$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\,\mathbf{w} \qquad \text{and} \qquad \mathbf{v_2} = \mathbf{v} - \mathbf{v_1}$$

Let's do the calculations.

$$\begin{array}{r} \mathbf{v} = \langle 1, -4\rangle \\ \cdot\ \mathbf{w} = \langle 2,\ \ 1\rangle \\ \hline \mathbf{v}\ \cdot\ \mathbf{w} = (1\cdot 2) + (-4\cdot 1) = -2 \end{array}$$

$$\|\mathbf{w}\|^{\mathbf{2}} = 2^2 + 1^2 = 5$$

Then,

$$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\,\mathbf{w} = \frac{-2}{5}\ \langle 2,\ \ 1\rangle = \left\langle -\frac{4}{5}, -\frac{2}{5}\right\rangle$$

$$\mathbf{v_1} = -\frac{\mathbf{4}}{\mathbf{5}}\mathbf{i}\ -\frac{\mathbf{2}}{\mathbf{5}}\mathbf{j}$$

And,

$$\begin{array}{r} \mathbf{v} = \langle 1, -4\rangle \\ +\ \ -\mathbf{v_1} = \langle \frac{4}{5},\ \ \frac{2}{5}\rangle \\ \hline \mathbf{v_2} = \mathbf{v} - \mathbf{v_1} = \langle \frac{9}{5}, \frac{-18}{5}\rangle \end{array} \qquad \mathbf{v_2} = \frac{\mathbf{9}}{\mathbf{5}}\mathbf{i}\ -\frac{\mathbf{18}}{\mathbf{5}}\mathbf{j}$$

Solve the problem.

41) A force is given by the vector **F** = 5**i** + 2**j**. The force moves an object along a straight line from the point (5, 7) to the point (18, 13). Find the work done if the distance is measured in feet and the force is measured in pounds.

Work is a scalar quantity in physics that measures the force exerted on an object over a particular distance. It is defined using vectors, as shown below. Let:

- $\mathbf{F}$ be the force vector acting on an object, moving it from point A to point B.
- $\overrightarrow{AB}$ be the vector from A to B.
- θ be the angle between $\mathbf{F}$ and $\overrightarrow{AB}$.

Then, we define work as:

$$\left.\begin{aligned} W &= \mathbf{F}\cdot\overrightarrow{AB} \\ W &= \|\mathbf{F}\|\,\left\|\overrightarrow{AB}\right\|\,\cos\theta \end{aligned}\right\}$$

Both of these formulas are useful. Which one you use in a particular situation depends on what information is available.

For this problem it is sufficient to use the first work formula, $W = \mathbf{F}\cdot\overrightarrow{AB}$

We are given $\mathbf{F} = \langle 5, 2\rangle$.

We can calculate $\overrightarrow{AB}$ as the difference between the two given points.

$$\begin{array}{r} (18, 13) \\ -\,(\ 5,\ \ 7) \\ \hline \overrightarrow{AB} = \langle 13,\ \ 6\rangle \end{array}$$

Note that the difference between two points is a vector.

Then, calculate $W = \mathbf{F}\cdot\overrightarrow{AB}$

$$\begin{array}{r} \mathbf{F} = \langle\ 5, 2\rangle \\ \cdot\ \overrightarrow{AB} = \langle 13, 6\rangle \\ \hline W = \mathbf{F}\cdot\overrightarrow{AB} = (5\cdot 13) + (2\cdot 6) = \textbf{77 foot pounds} \end{array}$$