A few things from the Calculus Handbook may be helpful (available for free at www.mathguy.us).

Limit Definition: The limit is the value $\boldsymbol{L}$ that a function approaches as the value of the input variable $\boldsymbol{x}$ approaches the desired value $\boldsymbol{a}$.

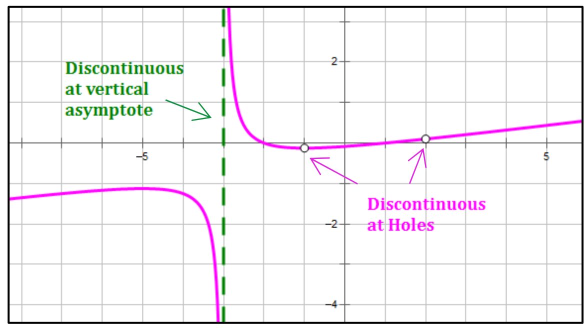
Limits may exist approaching $x = a$ from either the left $\left(\lim_{x \to a^-} f(x)\right)$ or the right $\left(\lim_{x \to a^+} f(x)\right)$. If the limits from the left and right are the same (e.g., they are both equal to L), then the limit exists at $x = a$ and we say $\lim_{x \to a} f(x) = L$.

Continuity Definition:

A function, f, is *continuous* at $x = c$ iff:

a. $f(c)$ is defined,
b. $\lim_{x \to c} f(x)$ exists, and
c. $\lim_{x \to c} f(x) = f(c)$
d. If $x = a$ is an endpoint, then the limit need only exist from the left or the right.

Basically, the function value and limit at a point must both exist and be equal to each other.



The curve shown is continuous everywhere except at the holes and the vertical asymptote.

Techniques for Finding Limits

Substitution

The easiest method, when it works, for determining a limit is **substitution.** Using this method, simply substitute the value of x into the limit expression to see if it can be calculated directly.

Example 1.1:

$$\lim_{x \to 3} \left(\frac{x+2}{x-2}\right) = \frac{3+2}{3-2} = \mathbf{5}$$

Simplification

When substitution fails, other methods must be considered. With rational functions (and some others), **simplification** may produce a satisfactory solution.

Example 1.2:

$$\lim_{x \to 5} \left(\frac{x^2-25}{x-5}\right) = \lim_{x \to 5} \left(\frac{(x+5)(x-5)}{(x-5)}\right) = x+5 = \mathbf{10}$$

Rationalization

Rationalizing a portion of the limit expression is often useful in situations where a limit is **indeterminate**. In the example below the limit expression has the indeterminate form $(-\infty + \infty)$. Other indeterminate forms are discussed later in this chapter.

Example 1.3:

$$\lim_{x \to -\infty} \left(x + \sqrt{x^2 - 8x} \right)$$

First, notice that this limit is taken to $-\infty$, which can often cause confusion. So, let's modify it so that we are taking the limit to $+\infty$. We do this using the substitution $x = -y$.

$$\lim_{x \to -\infty} \left(x + \sqrt{x^2 - 8x} \right) = \lim_{y \to +\infty} \left(-y + \sqrt{y^2 + 8y} \right)$$

Next, let's rationalize the expression in the limit by multiplying by a name for one, using its conjugate.

$$\lim_{y \to +\infty} \left(-y + \sqrt{y^2 + 8y} \right) = \lim_{y \to +\infty} \left(\frac{-y + \sqrt{y^2 + 8y}}{1} \cdot \frac{y + \sqrt{y^2 + 8y}}{y + \sqrt{y^2 + 8y}} \right)$$

$$= \lim_{y \to +\infty} \left(\frac{-y^2 + y^2 + 8y}{y + \sqrt{y^2 + 8y}} \right) = \lim_{y \to +\infty} \left(\frac{8y}{y + \sqrt{y^2 + 8y}} \right)$$

$$= \lim_{y \to +\infty} \left(\frac{8y}{y + \sqrt{y^2 + 8y}} \div \frac{y}{y} \right) = \lim_{y \to +\infty} \left(\frac{8}{1 + \sqrt{1 + \frac{8}{y}}} \right) = \frac{8}{1 + \sqrt{1}} = 4$$

L'Hospital's Rule

If f and g are differentiable functions and $g'(x) \neq 0$ near a and if:

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0 \quad \textbf{OR} \quad \lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty$$

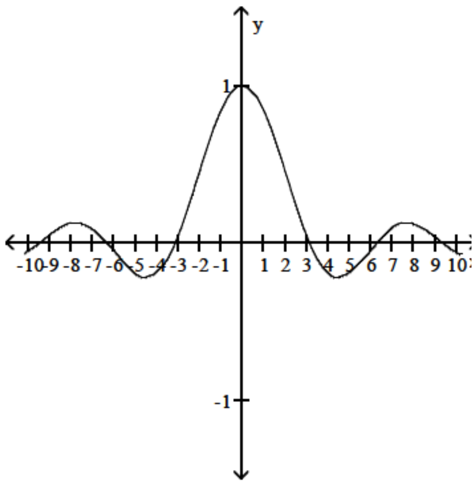
$$\text{Then,} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Note: L'Hospital's rule can be repeated as many times as necessary *as long as the result of each step is an indeterminate form*. If a step produces a form that is not indeterminate, the limit should be calculated at that point.

SHORT ANSWER. Write the word or phrase that best completes each statement or answers the question.

The graph of a function is given. Use the graph to find the indicated limit and finction value, or state that the limit or function value does not exist.

1) a. $\lim_{x \to 0} f(x)$ b. $f(0)$ 1) ______________



a) The limit of $f(x)$ at $x = 0$ exists if the limits from the left and right both exist and are equal. Clearly, $f(x)$ approaches the value of 1 from both the left and right. Therefore,

$$\lim_{x \to 0} f(x) = 1$$

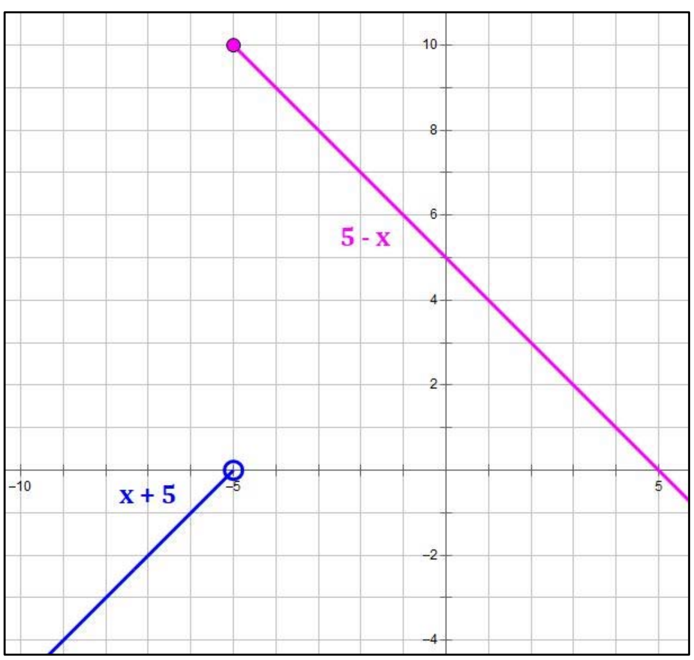
b) The value of $f(0)$ can be read from the graph:

$$f(0) = 1$$

Note that since the limits from the left and right of 0 both exist and are equal to $f(0)$, we can also conclude that **the function is continuous at $x = 0$.**

Graph the function. Then use your graph to find the indicated limit.

2) $f(x) = \begin{cases} x + 5 & x < -5 \\ 5 - x & x \geq -5 \end{cases}$, $\lim_{x \to -5} f(x)$ 2) ______________



The limits from the left and right are not the same:

$\lim_{x \to -5^-} f(x) = 0$ and $\lim_{x \to -5^+} f(x) = 10$.

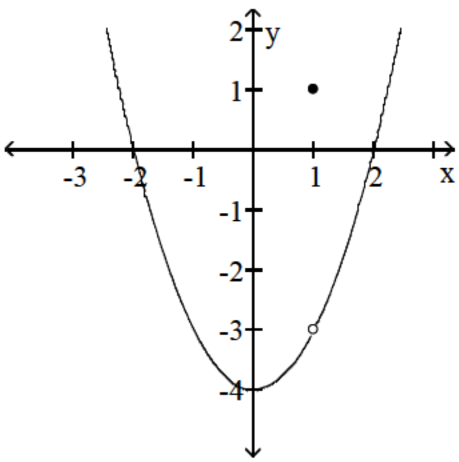
Since these limits are not the same,

$$\lim_{x \to -5} f(x) \text{ does not exist.}$$

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

The graph of a function is given. Use the graph to find the indicated limit and function value, or state that the limit or function value does not exist.

3) a. $\lim_{x \to 1} f(x)$ b. $f(1)$ 3) ________



The limits from the left and right both exist and are equal. Clearly, $f(x)$ approaches the value of -3 from both the left and right as x approaches 1. Therefore,

$$\lim_{x \to 1} f(x) = -3$$

The value of $f(1)$ can be read from the dot on the graph:

$$f(1) = 1$$ **Answer C**

Note that since the limit as x approaches 1 is not equal to $f(1)$, the function is NOT continuous at $x = 1$.

A) a. $\lim_{x \to 1} f(x) = 1$
b. $f(1) = -3$

B) a. $\lim_{x \to 1} f(x) = -3$
b. $f(1)$ does not exist

C) a. $\lim_{x \to 1} f(x) = -3$
b. $f(1) = 1$

D) a. $\lim_{x \to 1} f(x) = -3$
b. $(1) = -3$

SHORT ANSWER. Write the word or phrase that best completes each statement or answers the question.

Use properties of limits to find the indicated limit. It may be necessary to rewrite an expression before limit properties can be applied.

4) $\lim_{x \to 1} (x^2 - 2)^3$ 4) ________________

Polynomials are continuous everywhere, so limits exist for all values of x.

For a simple limit, try **substitution** first.

$$\lim_{x \to 1} (x^2 - 2)^3 = (1^2 - 2)^3 = (-1)^3 = \mathbf{-1}$$

5) $\lim_{x \to 5} \frac{x^2 - 2x - 15}{x + 3}$ 5) ________________

For a rational expression, try **simplification** first.

In this expression, there is a hole at $x = -3$. **In the case of a hole in a rational expression, a limit will exist but the function will not be continuous at the location of the hole.**

$$\lim_{x \to 5} \frac{x^2 - 2x - 15}{x + 3} = \lim_{x \to 5} \frac{(x+3)(x-5)}{x+3} = \lim_{x \to 5} (x - 5) = (5 - 5) = \mathbf{0}$$

6) $\lim_{x \to 1} \sqrt{3x - 2}$ 6) ______________

This function has an endpoint where $3x - 2 = 0$, i.e., at $x = \frac{2}{3}$. Since we are looking for a limit in the function's domain that is **NOT at an endpoint**, the limit will exist. **Limits do not exist at endpoints.**

Then, try **substitution** first.

$$\lim_{x \to 1} \sqrt{3x - 2} = \sqrt{3(1) - 2} = \sqrt{1} = \mathbf{1}$$

A piecewise function is given. Use the properties of limits to find the indicated limits, or state that the limit does not exist.

7) $f(x) = \begin{cases} -6x + 12 & \text{if } x < 1 \\ 5x + 1 & \text{if } x > 1 \end{cases}$ 7) ______________

a. $\lim_{x \to 1^-} f(x)$ b. $\lim_{x \to 1^+} f(x)$ c. $\lim_{x \to 1} f(x)$

a) From the left:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (-6x + 12) = (-6(1) + 12) = \mathbf{6}$$

b) From the right:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (5x + 1) = (5(1) + 1) = \mathbf{6}$$

c) Overall:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 6$$

Therefore, $\lim_{x \to 1} f(x)$ exists, and $\lim_{x \to 1} f(x) = \mathbf{6}$

8) $f(x) = \begin{cases} \dfrac{1}{x - 3} & \text{if } x > 3 \\ x^2 + 4x & \text{if } x \le 3 \end{cases}$ 8) ______________

a. $\lim_{x \to 3^-} f(x)$ b. $\lim_{x \to 3^+} f(x)$ c. $\lim_{x \to 3} f(x)$

a) From the left:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 + 4x) = (3^2 + 4(3)) = \mathbf{21}$$

b) From the right:

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \left(\frac{1}{x - 3}\right) \Rightarrow \left(\frac{1}{3^+ - 3}\right) = \boldsymbol{DNE}\ (\infty)$$

Note the use of the arrow instead of the equal sign when dealing with a limit that does not exist or with an indeterminate form (e.g., $\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{-\infty}$).

c) Overall:

If either the limit from the left or the limit from the right does not exist, the overall limit does not exist. Therefore, $\lim_{x \to 3} f(x)$ **does not exist** $(\boldsymbol{DNE})$

Use properties of limits to find the indicated limit. It may be necessary to rewrite an expression before limit properties can be applied.

9) $\lim\limits_{x \to -3} \dfrac{x^2 - 2x - 15}{x + 3}$ 9) ______________

For a rational expression, try **simplification** first. Note that there is a hole at $x = -3$.

$$\lim_{x \to -3} \frac{x^2 - 2x - 15}{x + 3} = \lim_{x \to -3} \frac{(x+3)(x-5)}{x+3} = \lim_{x \to -3} (x - 5) = (-3 - 5) = \mathbf{-8}$$

10) $\lim\limits_{x \to 2} \dfrac{\frac{1}{x} - \frac{1}{2}}{x - 2}$ 10) ______________

For a rational expression, try **simplification** first. Note that there is a hole at $x = 2$.

Let's start by getting a common denominator in the numerator of the given fraction.

$$\lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{(x-2)} = \lim_{x \to 2} \frac{\frac{2}{2} \cdot \frac{1}{x} - \frac{1}{2} \cdot \frac{x}{x}}{(x-2)} = \lim_{x \to 2} \frac{\frac{2-x}{2x}}{(x-2)} = \lim_{x \to 2} \frac{-(x-2)}{(x-2) \cdot 2x} = \lim_{x \to 2} \frac{-1}{2x} = \frac{-1}{2 \cdot 2} = -\mathbf{\frac{1}{4}}$$

11) $\lim\limits_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 4}$ 11) ______________

For a rational expression, try **simplification** first. Note that there is a hole at $x = 4$.

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{(x-4)} = \lim_{x \to 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{1}{(\sqrt{x} + 2)} = \frac{1}{(\sqrt{4} + 2)} = \mathbf{\frac{1}{4}}$$

Use the definition of continuity to determine whether f is continuous at a.

12) $f(x) = \dfrac{1}{x^2 - 6x}$ 12) ______________

$a = 6$

Recall that **for a function to be continuous at a point, the limit and the value of the function must both exist and must be equal at the desired value of x.** Let's start by looking at the limit of $f(x)$:

$$\lim_{x \to 6} \frac{1}{(x^2 - 6x)} \Rightarrow \frac{1}{(6^2 - 6(6))} \Rightarrow \frac{1}{0} \Rightarrow DNE$$

Since the limit does not exist at $x = 6$, we conclude that **$f(x)$ is not continuous at $x = 6$**.

13) $f(x) = \frac{x - 4}{x + 5}$ 13) ______________

$a = 4$

Recall that for a function to be continuous at a point, the limit and the value of the function must all exist and be equal at the desired value of x. Let's start by looking at the limit of $f(x)$:

$$\lim_{x \to 4} \left(\frac{x-4}{x+5}\right) = \left(\frac{4-4}{4+5}\right) = \frac{0}{9} = 0$$

Also,

$$f(4) = \left(\frac{4-4}{4+5}\right) = \frac{0}{9} = 0$$

Since the limit and function value both exist and are equal at $x = 4$, we conclude that **$f(x)$ is continuous at $x = 4$**.

Determine for what numbers, if any, the given function is discontinuous.

14) $f(x) = \begin{cases} x - 5 & \text{if } x \le 5 \\ x^2 - 10 & \text{if } x > 5 \end{cases}$ 14) ______________

To be discontinuous, the limits from the left and right need to be unequal. The only possible point of discontinuity is at $x = 5$, i.e., at the split between the two parts of the function.

a) From the left:

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (x - 5) = (5 - 5) = 0$$

b) From the right:

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (x^2 - 10) = 5^2 - 10 = 15$$

c) Overall:

Since $\lim_{x \to 5^-} f(x) \neq \lim_{x \to 5^+} f(x)$, $(\text{i.e.}, 0 \neq 15)$, we see that the limits are not equal from the left and the right. Therefore, we conclude that **$f(x)$ is not continuous at $x = 5$.**

Also, since $\lim_{x \to 5^-} f(x) \neq \lim_{x \to 5^+} f(x)$, we conclude that $\lim_{x \to 5^+} f(x)$ does not exist. Note that this is not required for the problem, but it's a good idea to see what else we can conclude from the problem.

15) $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ 15) ______________

Recall that for a function to be continuous at a point, the limit and the value of the function must both exist and be equal at the desired value of x. Let's start by looking at the limit of $f(x)$:

$$\lim_{x \to 3} \left(\frac{x^2 - 9}{x - 3} \right) = \lim_{x \to 3} \frac{(x+3)(x-3)}{(x-3)} = \lim_{x \to 3} (x+3) = 3 + 3 = 6$$

Also,

$f(3) = 6$ (from the second line of the definition of $f(x)$)

Since the limit and function value both exist and are equal at $x = 3$, we conclude that **$f(x)$ is continuous at $x = 3$**.

Find the slope of the tangent line to the graph of f at the given point.

16) $f(x) = x^2 + 5x$ at $(4, 36)$ 16) ______________

Slopes of tangent lines are obtained via the derivative of a function, evaluated at the point of tangency. We are required to use the limit definition of the derivative for this assignment.

Definition of a Derivative (from the Calculus Handbook)

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad \frac{d}{dx} f(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

So,

$f(x) = x^2 + 5x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^2 + 5(x+h)] - [x^2 + 5x]}{h}$$

$$= \lim_{h \to 0} \frac{[x^2 + 2hx + h^2 + 5x + 5h] - [x^2 + 5x]}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2 + 5h}{h} = \lim_{h \to 0} (2x + h + 5) = 2x + 5$$

$f'(x) = 2x + 5$

Substituting $x = 4$, we have:

$f'(4) = 2(4) + 5 = \mathbf{13}$

Find the slope-intercept equation of the tangent line to the graph of f at the given point.

17) $f(x) = 2x^2 + x - 3$ at $x = (4, 33)$ 17) ______________

The equation of a tangent line requires a point and a slope. We are given the point $(4, 33)$, but we need a slope. We are required to use the limit definition of the derivative for this assignment. So,

$$f(x) = 2x^2 + x - 3$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[2(x+h)^2 + (x+h) - 3] - [2x^2 + x - 3]}{h}$$

$$= \lim_{h\to 0}\frac{[2x^2 + 4hx + 2h^2 + x + h - 3] - [2x^2 + x - 3]}{h}$$

$$= \lim_{h\to 0}\frac{4hx + 2h^2 + h}{h} = \lim_{h\to 0}(4x + 2h + 1) = 4x + 1$$

$$f'(x) = 4x + 1$$

$f'(4) = 4(4) + 1 = 17$, which is the slope of the tangent line.

Then, using the slope and point we have for this problem, the equation of the tangent line is:

$y = 17(x - 4) + 33$ (in h, k form)

$y = 17x - 68 + 33$

$\boldsymbol{y = 17x - 35}$ (in slope-intercept form)

Use the Limit Definition to Find the derivative of f at x. That is, find f '(x).

18) $f(x) = x^2 - 8x - 11;\ x = 6$ 18) ______________

Presumably, we are being asked to use the first of the above limit definitions (the $x + h$ one), although at a given point, we could use either one.

$$f(x) = x^2 - 8x - 11$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[(x+h)^2 - 8(x+h) - 11] - [x^2 - 8x - 11]}{h}$$

$$= \lim_{h\to 0}\frac{[x^2 + 2hx + h^2 - 8x - 8h - 11] - [x^2 - 8x - 11]}{h}$$

$$= \lim_{h\to 0}\frac{2hx + h^2 - 8h}{h} = \lim_{h\to 0}(2x + h - 8) = 2x - 8$$

$\boldsymbol{f'(x) = 2x - 8}$

$f'(6) = 2\cdot 6 - 8 = \mathbf{4}$

19) $f(x) = 7x + 8;\ x = 5$ 19) ______________

$$f(x) = 7x + 8$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[7(x+h)+8]-[7x+8]}{h}$$

$$= \lim_{h\to 0}\frac{[7x+7h+8]-[7x+8]}{h}$$

$$= \lim_{h\to 0}\frac{7h}{h} = \lim_{h\to 0}(7) = 7$$

$$\mathbf{f'(x) = 7}$$

$f'(5) = \mathbf{7}$ In fact, $f'(\text{any value of } x) = 7$ (note that $f(x) = 7x + 8$ is a line with slope 7)

20) $f(x) = \sqrt{x};\ x = 100$ 20) ______________

$$f(x) = \sqrt{x}$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0}\frac{(x+h)-x}{h\cdot\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0}\frac{h}{h\cdot\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0}\frac{1}{\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \frac{1}{\left(\sqrt{x}+\sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$$

$$\mathbf{f'(x) = \frac{1}{2\sqrt{x}}}$$

$$f'(100) = \frac{1}{2\sqrt{100}} = \mathbf{\frac{1}{20}}$$

Solve the problem.

21) The function $f(x) = x^3$ describes the volume of a cube, $f(x)$, in cubic inches, whose length, width, and height each measure x inches. If x is changing, find the instantaneous rate of change of the volume with respect to x at the moment when $x = 2$ inches. 21) ______________

We are given: $f(x) = x^3$ $\quad x = 2$

"Instantaneous rate of change" is wording for the derivative. So, we are being asked for $f'(2)$.

We are required to use the limit definition of the derivative for this assignment. So,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[(x+h)^3]-[x^3]}{h}$$

$$= \lim_{h\to 0}\frac{[x^3+3x^2h+3xh^2+h^3]-[x^3]}{h}$$

$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2$$

$f'(x) = 3x^2$

$f'(2) = 3\cdot 2^2 =$ **12 inches³ per inch of length, width, or height**

Note: The units in this problem are weird, but word problems typically require units. Note that, in Calculus class, there would typically be a time element in this kind of problem, so the units would be "per minute", "per second", or something similar.

22) A foul tip of a baseball is hit straight upward from a height of 4 feet with an initial velocity of 96 feet per second. The function $s(t) = -16t^2 + 96t$ describes the ball's height above the ground, s(t), in feet, t seconds after it was hit. What is the instantaneous velocity of the ball 2.1 seconds after it was hit? 22) ______________

$s(t) = -16t^2 + 96t + 4$ This is the correct position function (not the one given).

Recall that **instantaneous velocity is the derivative of position**.

$$v(t) = s'(t) = \lim_{h\to 0}\frac{s(t+h)-s(t)}{h} = \lim_{h\to 0}\frac{[-16(t+h)^2+96(t+h)+4]-[-16t^2+96t+4]}{h}$$

$$= \lim_{h\to 0}\frac{[-16t^2-32ht-16h^2+96t+96h+4]-[-16t^2+96t+4]}{h}$$

$$= \lim_{h\to 0}\frac{-32ht-16h^2+96h}{h} = \lim_{h\to 0}(-32t-16h+96) = -32t+96$$

$v(t) = s'(t) = -32t + 96$ This is the velocity function.

$v(2.1) = -32\cdot(2.1) + 96 =$ **28.8 ft. per second.**