For #1 – 4, solve each system of equations by using the method of your choice.

1) $\begin{cases} x - y + z = -1 \\ x + y + z = -9 \\ x + y - z = -3 \end{cases}$

In most cases, you begin solving a set of three equations by selecting pairs of equations and eliminating the same variable in each pair. I'll start this one by eliminating the variable z.

Add 1st and 3rd equations

$$\begin{array}{l} x - y + z = -1 \\ x + y - z = -3 \\ \hline 2x \qquad\quad = -4 \end{array}$$

Add 2nd and 3rd equations

$$\begin{array}{l} x + \ y + z = \ -9 \\ x + \ y - z = \ -3 \\ \hline 2x + 2y \qquad = -12 \end{array}$$

We got lucky here because we were able to eliminate two variables at the same time by adding the first and third equations. Normally, this would not happen, and you would have to solve the set of two simultaneous equations which result. Let's continue.

Solve for x:

$2x = -4$

$x = -2$

Then, solve for y:

$2x + 2y = -12$

$2 \cdot (-2) + 2y = -12$

$-4 + 2y = -12$

$2y = -8$

$y = -4$

Then, solve for z:

$x - y + z = -1$

$(-2) - (-4) + z = -1$

$2 + z = -1$

$z = -3$

Finally, test your results in one of the original equations, but not the one used to solve for the third variable solved for above. In this case, we used the 1st equation to solve for our final variable, z. So, we should use either the 2nd or 3rd equation to check our answer.

Second equation: $(-2) + (-4) + (-3) = -9$ ✓

Solution: $(-2, -4, -3)$

2) $\begin{cases} x + y = 6 \\ y = x^2 - 8x + 16 \end{cases}$

$x + y = 6 \qquad y = x^2 - 8x + 16$

$x + (x^2 - 8x + 16) = 6$

$x^2 - 7x + 10 = 0$

$(x - 5)(x - 2) = 0$

$x = \{2, 5\}$

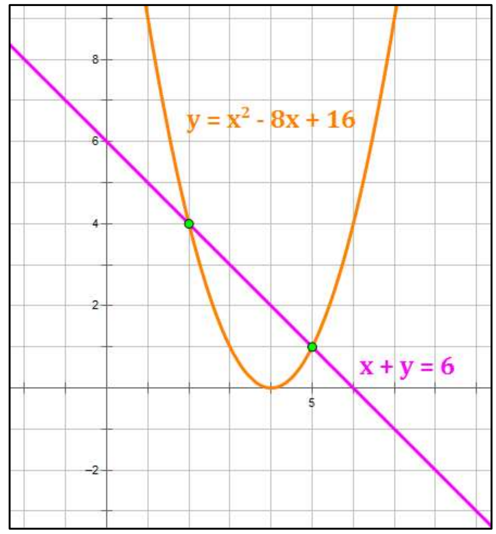
When $x = 2$, we get:

$2 + y = 6$, so $y = 4 \Rightarrow (2, 4)$ is a solution

When $x = 5$, we get:

$5 + y = 6$, so $y = 1 \Rightarrow (5, 1)$ is a solution

So, our solutions are: $\{(\mathbf{2}, \mathbf{4}), (\mathbf{5}, \mathbf{1})\}$



3) $\begin{cases} x + y = -15 \\ xy = 56 \end{cases}$

$x + y = -15 \qquad xy = 56$

$y = -15 - x \qquad x(-15 - x) = 56$

$-x^2 - 15x = 56$

$x^2 + 15x + 56 = 0$

$(x + 7)(x + 8) = 0$

$x = \{-7, -8\}$

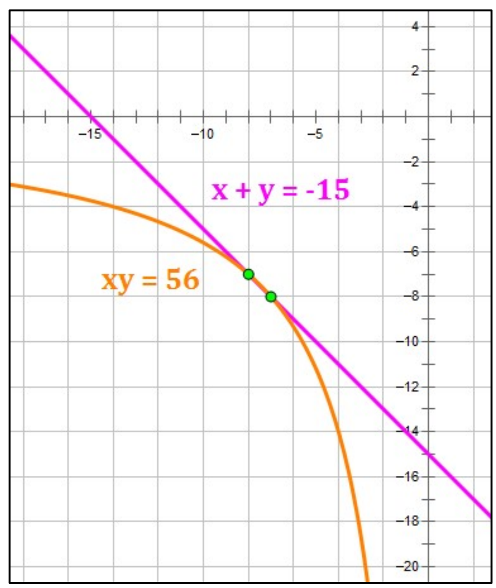
When $x = -7$, we get:

$y = -15 - (-7) = -8 \Rightarrow (-7, -8)$ is a solution

When $x = -8$, we get:

$y = -15 - (-8) = -7 \Rightarrow (-8, -7)$ is a solution

So, our solutions are: $\{(\mathbf{-7}, \mathbf{-8}), (\mathbf{-8}, \mathbf{-7})\}$



4) $\begin{cases} 2x^2 + y^2 = 66 \\ x^2 + y^2 = 41 \end{cases}$

$2x^2 + y^2 = 66 \qquad x^2 + y^2 = 41$

Let's use the Addition (i.e., Elimination) Method

$2x^2 + y^2 = 66$	multiply by (1)	$2x^2 + y^2 = \ \ 66$
$x^2 + y^2 = 41$	multiply by (-1)	$-x^2 - y^2 = -41$
		$x^2 \qquad = \ \ 25$
		$x = \pm 5$

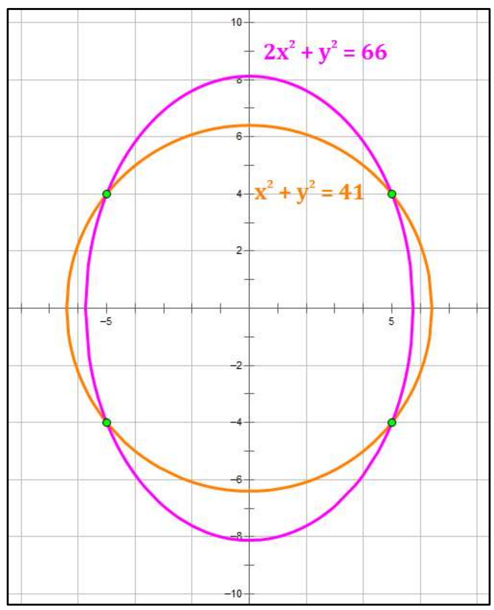
When $x = 5$, we get:

$5^2 + y^2 = 41$

$25 + y^2 = 41$

$y^2 = 16 \quad \Rightarrow \quad y = \pm 4$

$(5, 4)$ and $(5, -4)$ are solutions



When $x = -5$, we get:

$(-5)^2 + y^2 = 41$

$25 + y^2 = 41$

$y^2 = 16 \quad \Rightarrow \quad y = \pm 4$

$(-5, 4)$ and $(-5, -4)$ are solutions

So, the entire solution set is:

$\{(\mathbf{5}, \mathbf{4}), (\mathbf{5}, -\mathbf{4}), (-\mathbf{5}, \mathbf{4}), (-\mathbf{5}, -\mathbf{4})\}$

For problems 5 to 7, we need only write the form of the decomposition. We do not need to solve the decomposition for the constants $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$.

Write the form of the partial fraction decomposition of the rational expression. It is not necessary to solve for the constants.

5) $\frac{2x+3}{(x-6)(x+6)}$

This rational function has no repeated linear factors in the denominator, so the decomposition is straightforward:

$$\frac{2x+3}{(x-6)(x+6)} = \frac{\boldsymbol{A}}{\boldsymbol{x-6}} + \frac{\boldsymbol{B}}{\boldsymbol{x+6}}$$

6) $\frac{(3x-1)}{(x+5)(x+7)^2}$

This rational function has a repeated linear factor $(x+7)$ in the denominator. The decomposition must include terms with each integer exponent of $(x+7)$, up to the exponent of $(x+7)$ in the denominator of the rational function. In this case the exponent is 2, so we include terms with $(x+7)$ and $(x+7)^2$ in the decomposition:

$$\frac{3x-1}{(x+5)(x+7)^2} = \frac{\boldsymbol{A}}{\boldsymbol{x+5}} + \frac{\boldsymbol{B}}{\boldsymbol{x+7}} + \frac{\boldsymbol{C}}{\boldsymbol{(x+7)^2}}$$

7) $\frac{2x-5}{(x+2)(x^2+x-4)}$

This rational function has a quadratic function in the denominator so the decomposition must take this into account:

$$\frac{2x-5}{(x+2)(x^2+x-4)} = \frac{\boldsymbol{A}}{\boldsymbol{x+2}} + \frac{\boldsymbol{Bx+C}}{\boldsymbol{x^2+x-4}}$$

For #8 – 10, write the partial fraction decomposition of each rational expression.

8) $\frac{15x-39}{(x-1)(x-5)}$

Write the form of the decomposition: $\frac{15x-39}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5}$

Multiply both sides by $(x-1)(x-5)$: $15x - 39 = A(x-5) + B(x-1)$

Simplify: $15x - 39 = (A + B)x + (-5A - B)$

Write the simultaneous equations and solve them:

$A + B = 15$ $\quad -5A - B = -39$

Solve for A:

$$A + B = 15$$
$$-5A - B = -39$$
$$-4A = -24$$
$$\boldsymbol{A = 6}$$

Then, solve for B:

$$A + B = 15$$
$$6 + B = 15$$
$$\boldsymbol{B = 9}$$

So, the partial fraction decomposition is:

$$\frac{15x-39}{(x-1)(x-5)} = \frac{\mathbf{6}}{\boldsymbol{x-1}} + \frac{\mathbf{9}}{\boldsymbol{x-5}}$$

9) $\frac{36-7x}{x(x-3)^2}$

Write the form of the decomposition: $\frac{-7x+36}{x(x-3)^2} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$

Multiply both sides by $x(x-3)^2$: $-7x + 36 = A(x-3)^2 + Bx(x-3) + Cx$

Expand and simplify: $-7x + 36 = A(x^2 - 6x + 9) + B(x^2 - 3x) + Cx$

$-7x + 36 = (A + B)x^2 + (-6A - 3B + C)x + 9A$

Write the simultaneous equations and solve them:

$A + B = 0$ $\quad -6A - 3B + C = -7$ $\quad 9A = 36$

Solve for A:

$9A = 36$

$\mathbf{A = 4}$

Then, solve for B:

$A + B = 0$

$4 + B = 0$

$\mathbf{B = -4}$

Then, solve for C:

$-6A - 3B + C = -7$

$-6(4) - 3(-4) + C = -7$

$-24 + 12 + C = -7$

$\mathbf{C = 5}$

So, the partial fraction decomposition is:

$$\frac{-7x + 36}{x(x-3)^2} = \frac{4}{x} + \frac{-4}{x-3} + \frac{5}{(x-3)^2}$$

10) $\frac{10x+2}{(x-1)(x^2+x+1)}$

Write the form of the decomposition: $\frac{10x+2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

Multiply both sides by $(x-1)(x^2+x+1)$: $10x+2 = A(x^2+x+1) + (Bx+C)(x-1)$

Expand and simplify: $10x+2 = (Ax^2+Ax+A) + (Bx^2-Bx+Cx-C)$

$$10x+2 = (A+B)x^2 + (A-B+C)x + (A-C)$$

Write the simultaneous equations and solve them:

$A+B=0$ $\quad$ $A-B+C=10$ $\quad$ $A-C=2$

Now, let's eliminate C form the 2nd equation with some help from the 3rd equation.

$$\begin{array}{r} A-B+C=10 \\ A\qquad -C=\ \ 2 \\ \hline 2A-B\qquad =12 \end{array}$$

Then,

<u>Solve for A:</u>

$$\begin{array}{r} 2A-B=12 \\ A+B=\ \ 0 \\ \hline 3A\qquad =12 \end{array}$$

$$\boldsymbol{A=4}$$

<u>Then, solve for B:</u>

$A+B=0$

$4+B=0$

$\boldsymbol{B=-4}$

<u>Then, solve for C:</u>

$A-C=2$

$4-C=2$

$\boldsymbol{C=2}$

So, the partial fraction decomposition is:

$$\frac{10x+2}{(x-1)(x^2+x+1)} = \frac{\mathbf{4}}{\boldsymbol{x-1}} + \frac{\boldsymbol{-4x+2}}{\boldsymbol{x^2+x+1}}$$

11) Write the partial fraction decomposition for $\frac{x^2+3x+1}{(x^2+4)^2}$.

Write the form of the decomposition: $\frac{x^2+3x+1}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$

Multiply both sides by $(x^2+4)^2$: $x^2+3x+1 = (Ax+B)(x^2+4) + Cx + D$

Expand and simplify: $x^2+3x+1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx + D$

$x^2+3x+1 = Ax^3 + Bx^2 + (4A+C)x + (4B+D)$

Write the simultaneous equations and solve them:

$A = 0$ $B = 1$ $4A + C = 3$ $4B + D = 1$

Solve for C:

$4A + C = 3$

$4(0) + C = 3$

$\boldsymbol{C = 3}$

Then, solve for D:

$4B + D = 1$

$4(1) + D = 1$

$4 + D = 1$

$\boldsymbol{D = -3}$

So, the partial fraction decomposition is:

$$\frac{x^2+3x+1}{(x^2+4)^2} = \frac{\mathbf{1}}{\boldsymbol{x^2+4}} + \frac{\boldsymbol{3x-3}}{\boldsymbol{(x^2+4)^2}}$$

12) Solve by any method of your choice: $\begin{cases} x^2 + y^2 = 29 \\ 4x + y^2 = 17 \end{cases}$

$x^2 + y^2 = 29 \qquad 4x + y^2 = 17$

Let's use the Addition (i.e., Elimination) Method

$x^2 + y^2 = 29$ multiply by (1)

$4x + y^2 = 17$ multiply by (-1)

$$\begin{array}{rcr} x^2 \quad + y^2 &=& 29 \\ -4x - y^2 &=& -17 \\ \hline x^2 - 4x \qquad &=& 12 \end{array}$$

$$x^2 - 4x - 12 = 0$$

$$(x + 2)(x - 6) = 0$$

$$x = \{-2, 6\}$$

When $x = -2$, we get:

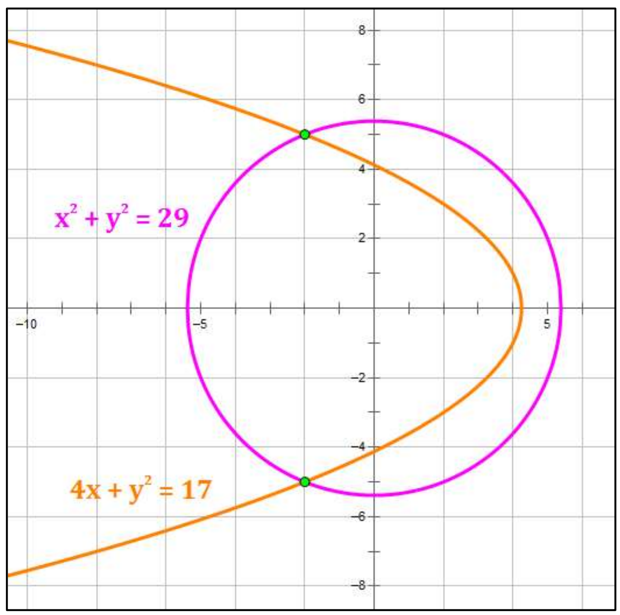
$4x + y^2 = 17$

$4(-2) + y^2 = 17$

$-8 + y^2 = 17$

$y^2 = 25 \quad \Rightarrow \quad y = \pm 5$

$(-2, 5)$ and $(-2, -5)$ are solutions



When $x = 6$, we get:

$4x + y^2 = 17$

$4(6) + y^2 = 17$

$24 + y^2 = 17$

$y^2 = -7$

$y = \pm i\sqrt{7}$

$(6, i\sqrt{7})$ and $(6, -i\sqrt{7})$ are complex solutions

So, the entire solution set is:

$\{(\mathbf{-2, 5}), (\mathbf{-2, -5}), (\mathbf{6}, \boldsymbol{i}\sqrt{\mathbf{7}}), (\mathbf{6}, -\boldsymbol{i}\sqrt{\mathbf{7}})\}$

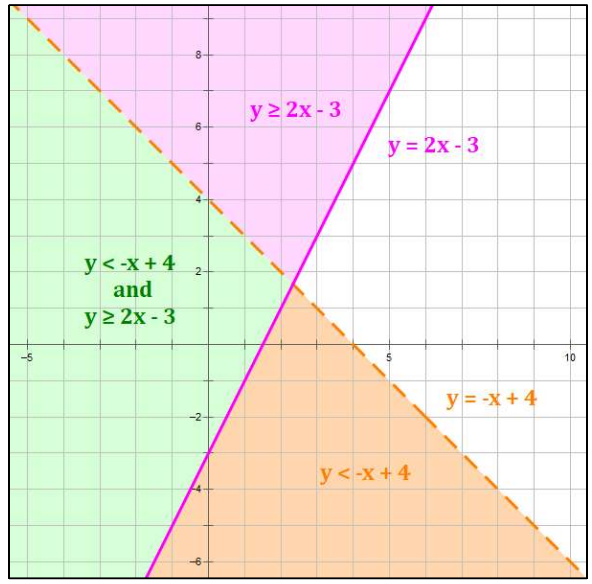
Note, if we are looking for only real solutions, e.g., when we are graphing, the solution set would be only $\{(-2,5), (-2,-5)\}$.

For #13 – 16, graph each solution set of the system of inequalities, or indicate that there is no solution.

13) $\begin{cases} y < -x + 4 \\ y \geq 2x - 3 \end{cases}$

$\boldsymbol{y < -x + 4}$ (orange and green areas)

- Graph the line: $y = -x + 4$.
- The line will be dashed because there is no "equal sign" included in the inequality.
- Fill in the portion of the graph below the line because of the "less than" portion of the inequality.



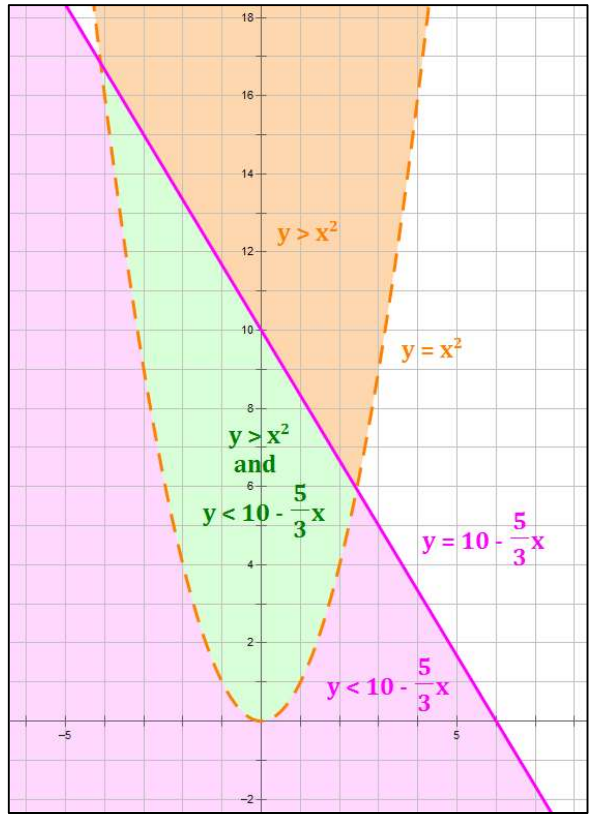
$\boldsymbol{y \geq 2x - 3}$ (violet and green areas)

- Graph the line: $y = 2x - 3$.
- The line will be solid because there is an "equal sign" included in the inequality.
- Fill in the portion of the graph above the line because of the "greater than" portion of the inequality.

The green area is the area of intersection of the two linear inequalities.

14) $\begin{cases} y > x^2 \\ 10x + 6y \leq 60 \end{cases}$

$\boldsymbol{y > x^2}$ (orange and green areas)



- Graph the parabola: $y = x^2$.
- Some points on the parabola: $(0,0), (2,4), (-2,4)$
- The parabola will be dashed because there is no "equal sign" included in the inequality.
- Fill in the portion of the graph above the parabola because of the "greater than" in the inequality.

$\mathbf{10x + 6y \leq 60}$ (violet and green areas)

- Put this in slope intercept form

  $10x + 6y \leq 60$

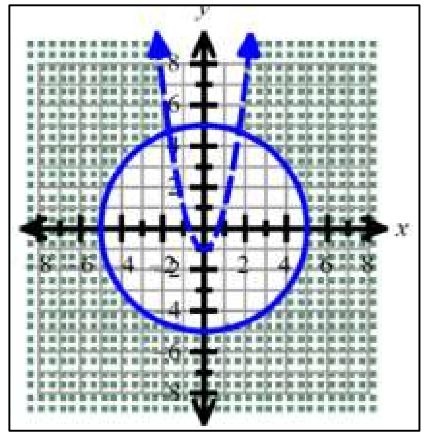
  $6y \leq 60 - 10x$

  $y \leq 10 - \frac{5}{3}x$
- Graph the line: $y = 10 - \frac{5}{3}x$.
- The line will be solid because there is an "equal sign" included in the inequality.
- Fill in the portion of the graph below the line because of the "less than" portion of the inequality.

The green area is the area of intersection of the given inequalities.

15) $\begin{cases} x^2 + y^2 \geq 25 \\ y - 2x^2 < -1 \end{cases}$

$\boldsymbol{x^2 + y^2 \geq 25}$



- Graph the circle: $x^2 + y^2 = 25$.
- Some points on the circle: $(0,5), (0,-5), (5,0), (-5,0)$
- The circle will be solid because there is an "equal sign" included in the inequality.
- The area desired is outside of the circle because of the "greater than" portion of the inequality.

$\boldsymbol{y - 2x^2 < -1}$

Put this in "$y <$" form

$y - 2x^2 < -1$

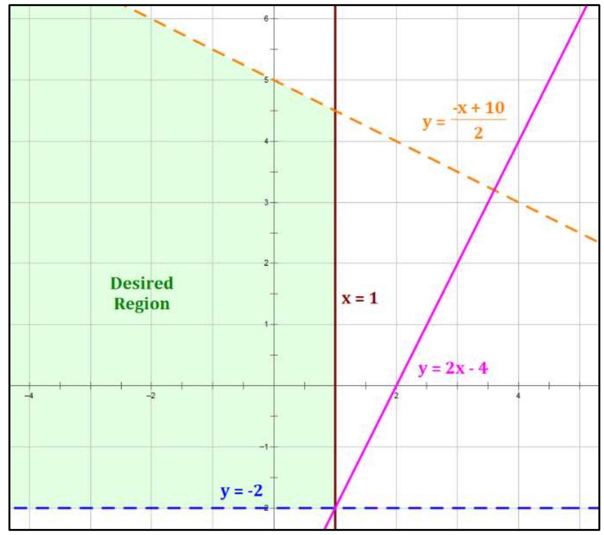
$y < 2x^2 - 1$

- Graph the parabola: $y = 2x^2 - 1$.
- Some points on the parabola: $(0,-1), (2,7), (-2,7)$
- The parabola will be dashed because there is no "equal sign" included in the inequality.
- Fill in the portion of the graph below the parabola because of the "less than" portion of the inequality.

Finally, bringing the two inequalities together, we want the area outside the circle that is also below (outside of) the parabola.

The shaded area is the area of intersection of the given inequalities.

16) $\begin{cases} y \geq 2x - 4 \\ x + 2y < 10 \\ y > -2 \\ x \leq 1 \end{cases}$



Let's graph the equations associated with each inequality and look at the desired region. The inequalities are:

$y \geq 2x - 4$

$x + 2y < 10 \quad \Rightarrow \quad y < \dfrac{-x + 10}{2}$

$y > -2$

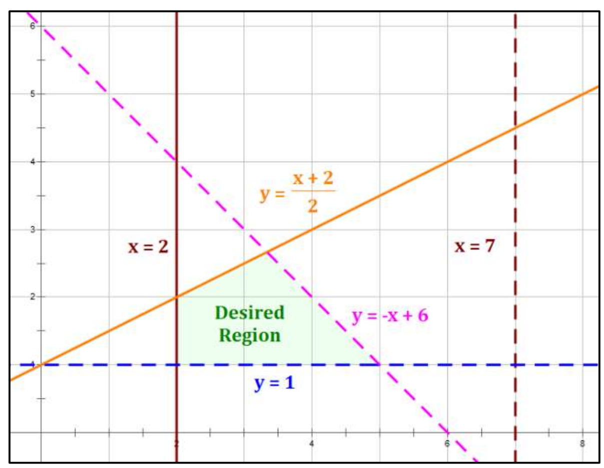
$x \leq 1$

The magenta and brown lines are solid because each inequality contains an equal sign. The orange and blue lines are dashed because there is no "equal sign" included in the inequalities.

Notice that, for this problem, the restriction $y \geq 2x - 4$ has no impact on the region created.

The green area is the area of intersection of the given inequalities.

17) $\begin{cases} y > 1 \\ 2 \leq x < 7 \\ x - 2y \geq -2 \\ x + y < 6 \end{cases}$



Let's graph the equations associated with each inequality and look at the desired region. The inequalities are:

$y > 1$

$2 \leq x < 7 \quad \Rightarrow \quad x \geq 2 \text{ and } x < 7$

$x - 2y \geq -2 \quad \Rightarrow \quad y \leq \dfrac{x + 2}{2}$

$x + y < 6 \quad \Rightarrow \quad y < -x + 6$

The magenta, blue, and $x = 7$ lines are dashed because there is no "equal sign" included in the inequalities.

The orange and $x = 2$ lines are solid because each inequality contains an equal sign.

Notice that, for this problem, the restriction $x < 7$ has no impact on the region created.

The green area is the area of intersection of the given inequalities.

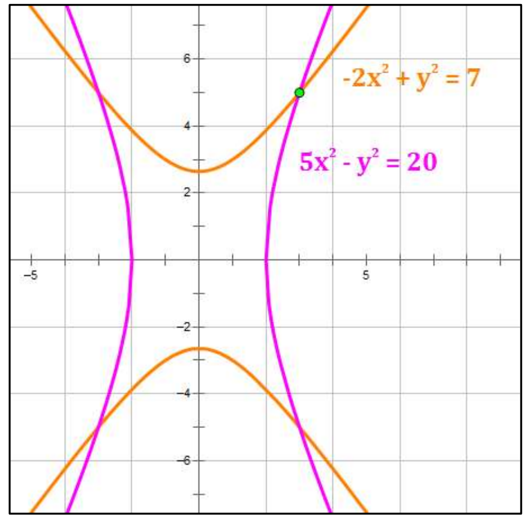
Solve the problem.

18) A system for tracking ships indicated that a ship lies on a hyperbolic path described by $5x^2 - y^2 = 20$. The process is repeated and the ship is found to lie on a hyperbolic path described by $y^2 - 2x^2 = 7$. If it is known that the ship is located in the first quadrant of the coordinate system, determine its exact location.

Let's use the Addition (i.e., Elimination) Method

$$\begin{aligned} 5x^2 - y^2 &= 20 \\ -2x^2 + y^2 &= \ \ 7 \\ \hline 3x^2 \qquad &= 27 \end{aligned}$$

$$x = \{\pm 3\}$$



Since the ship is in Q1, we must have positive x and a positive y. So, $x = 3$. Then,

$5x^2 - y^2 = 20$

$5(3)^2 - y^2 = 20$

$45 - y^2 = 20$

$y^2 = 25 \quad \Rightarrow \quad y = 5$ in Q1 So, **$(3, 5)$ is the exact location**

Let x represent one number and let y represent the other number. Use the given conditions to write a system of nonlinear equations. Solve the system and find the numbers.

19) The sum of two numbers is -7 and their product is -144. Find the numbers.

$x + y = -7 \qquad xy = -144$

Let's use the Substitution Method

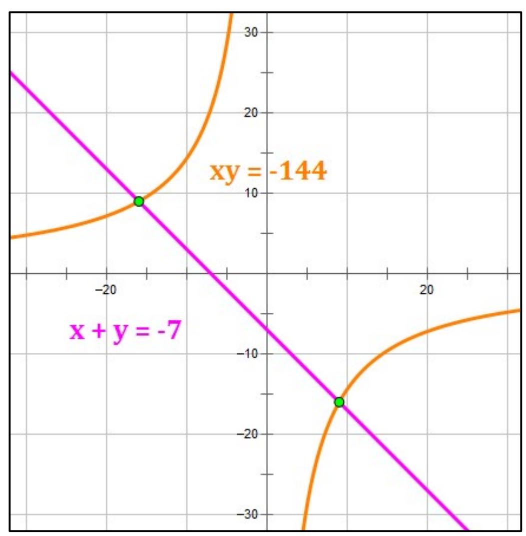
$y = -7 - x \qquad x(-7 - x) = -144$

$-x^2 - 7x = -144$

$x^2 + 7x - 144 = 0$

$(x + 16)(x - 9) = 0$

$x = \{-16, 9\}$



When $x = -16$, we get:

$y = -7 - (-16) = 9 \Rightarrow (-16, 9)$ is a solution

When $x = 9$, we get:

$y = -7 - (9) = -16 \Rightarrow (9, -16)$ is a solution

So, the two numbers are: **-16 and 9**

20) The sum of the squares of two numbers is 37. The sum of the two numbers is -5. Find the two numbers.

$x^2 + y^2 = 37$ $\quad$ $x + y = -5$

Let's use the Substitution Method

$y = -x - 5$ $\quad$ $x^2 + (-x - 5)^2 = 37$

$x^2 + (x^2 + 10x + 25) = 37$

$2x^2 + 10x - 12 = 0$

$x^2 + 5x - 6 = 0$

$(x + 6)(x - 1) = 0$

$x = \{-6, 1\}$

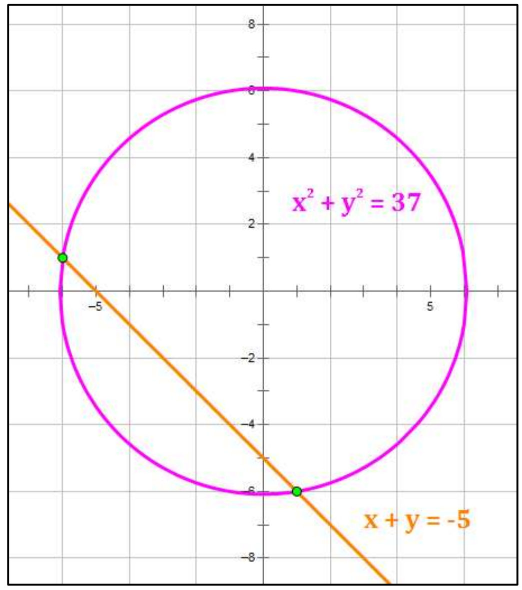
When $x = -6$, we get:

$y = -(-6) - 5 = 1 \Rightarrow (-6, 1)$ is a solution

When $x = 1$, we get:

$y = -(1) - 5 = -6 \Rightarrow (1, -6)$ is a solution

So, the two numbers are: **-6 and 1**



For #21 – 23, solve each problem by using a system of equations.

21) Find the dimensions of a rectangle with perimeter of 42 ft and area of 90 ft^2.

Let the dimensions of the rectangle be: x by y. Then, $P = 2x + 2y = 42$ $\quad$ $A = xy = 90$

Let's use the Substitution Method

From P, we get: $y = 21 - x$

Then, from A, we get: $x(21 - x) = 90$

$-x^2 + 21x = 90$

$x^2 - 21x + 90 = 0$

$(x - 15)(x - 6) = 0$

$x = \{6, 15\}$

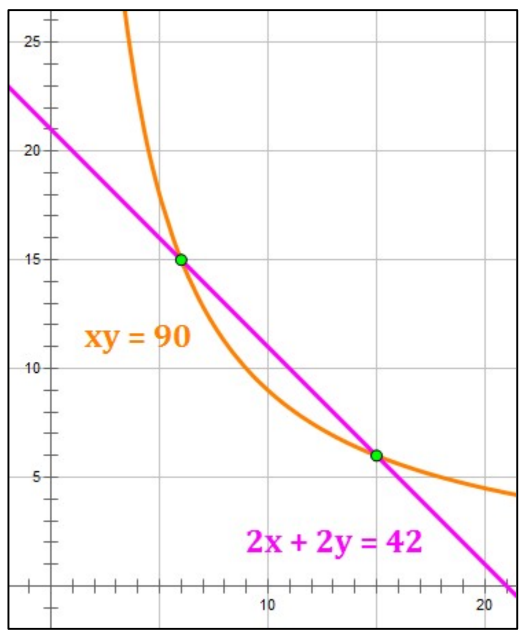
When $x = 6$, we get:

$y = 21 - (6) = 15 \Rightarrow (6, 15)$ is a solution

When $x = 15$, we get:

$y = 21 - (15) = 6 \Rightarrow (15, 6)$ is a solution

So, the dimensions are: **15** ft. by **6** ft.



22) In a chemistry class, 9 liters of a 4% silver iodide solution must be mixed with a 10% solution to get a 6% solution. How many liters of the 10% solution are needed?

9 liters of 4% solution plus x liters of 10% solution yields $(9 + x)$ liters of 6% solution.

$$9(0.04) + x(0.10) = (9 + x)(0.06)$$

Multiply by 100 on both sides to get rid of the decimals.

$$9 \cdot 4 + 10x = 6(9 + x)$$

$$36 + 10x = 54 + 6x$$

$$4x = 18$$

$x = 4.5$ liters of 10% solution are needed.

23) A chemist needs 170 milliliters of a 27% solution but only has 17% and 51% solutions available. How many milliliters of each should be mixed to obtain the desired solution?

x milliliters of 17% solution $+$ y milliliters of 51% solution $=$ 170 milliliters of 27% solution.

$$x + y = 170 \quad \Rightarrow \quad y = 170 - x$$

$$x(0.17) + y(0.51) = 170(0.27)$$

Substitute $170 - x$ for y in the above equation.

$$x(0.17) + (170 - x)(0.51) = 170(0.27)$$

Multiply by 100 on both sides to get rid of the decimals.

$$17x + 51(170 - x) = 27 \cdot 170$$

$$17x + 8670 - 51x = 4590$$

$$-34x + 8670 = 4590$$

$$-34x = -4080$$

$x = 120$ mL of 17% solution are needed.

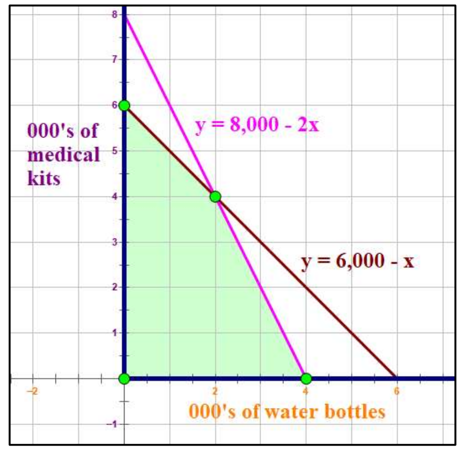
$y = 170 - x = 50$ mL of 51% solution are needed.

24) Bottled water and medical supplies are to be shipped to survivors of an earthquake by plane. Each container of water bottles will serve 10 people, and each medical kit will aid 6 people. Each plane can carry no more than 80,000 pounds. The bottled water weighs 20 pounds per container, and each medical kit weighs 10 pounds. Each plane can carry a total volume of supplies that does not exceed 6000 ft^3. Each water bottle is 1 ft^3, as is each medical kit. Determine how many bottles of water and how many medical kits should be sent on each plane to maximize the number of earthquake survivors who can be helped.

Let: x = number of water bottles

Let: y = number of medical kits

Objective function: $10x + 6y$



Constraints:

$20x + 10y \le 80{,}000 \;\Rightarrow\; y \le 8{,}000 - 2x$

$x + y \le 6{,}000 \;\Rightarrow\; y \le 6{,}000 - x$

$x \ge 0$

$y \ge 0$

These two constraints place the entire region in Quadrant 1.

Point of intersection:

$8{,}000 - 2x = 6{,}000 - x$

$2{,}000 = x$

$y = 6{,}000 - 2{,}000 = 4{,}000$

Point $= (2000, 4000)$

Points at the vertices of the bounded region:

$(0, 0)$

$(0, 6000)$

$(4000, 0)$

$(2000, 4000)$

Objective function evaluated at each vertex of the bounded region (points are green in the above illustration):

$(0, 0)$	$10x + 6y = 0$
$(0, 6000)$	$10x + 6y = 10(0) + 6(6000) = 36{,}000$
$(4000, 0)$	$10x + 6y = 10(4000) + 6(0) = 40{,}000$
$(2000, 4000)$	$10x + 6y = 10(2000) + 6(4000) = 44{,}000$ ✓ (largest number)

Solution: 2000 water bottles and 4000 medical kits should be sent on each plane.

Note: this problem is conflating water bottles and water bottle containers. So we must assume that there is one water bottle per water bottle container to obtain the desired solution. Note also that each plane will provide water for 2,000 x 10 = 20,000 people and medical kits to serve 4,000 x 6 = 24,000 people. That seems a little odd.