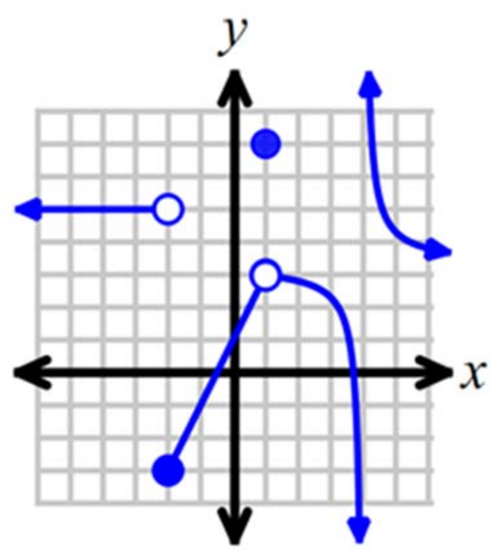
Ch 11 Review Worksheet: Do all work on your own paper. Calculator allowed.

A lot of information about limits, derivatives, and other Calculus topics is available in the Calculus Handbook at www.mathguy.us. You may find this resource useful in studying Calculus.

Informal Definition: The limit is the value L that a function approaches as the value of the input variable x approaches the desired value a.

Limits may exist approaching $x = a$ from either the left, $\lim_{x \to a^-} f(x)$, or the right, $\lim_{x \to a^+} f(x)$. If the limits from the left and right are the same (e.g., they are both equal to L), then the limit exists at $x = a$ and we say $\lim_{x \to a} f(x) = L$.

For #1 – 10: Use the graph of $h(x)$ below to find the requested values, if possible. **#1 – 9: Multiple Choice.**



1) $h(-2)$

A) 5 (B) -3 C) -2 D) DNE

This question asks for the value of the function @ $x = -2$. Note that the curve is solid at the point $(-2, -3)$. Therefore, $\boldsymbol{h(-2) = -3.}$

Answer B

2) $\lim_{x \to -2} h(x)$

A) 5 B) -3 C) -2 (D) DNE

The limits from the left and right are different @ $x = -2$. From the left, the curve approaches $y = 5$, and from the right, the curve approaches $y = -3$. Therefore, **the limit does not exist.**

Answer D

3) $h(1)$

(A) 7 B) 1 C) 3 D) DNE

This question asks for the value of the function @ $x = 1$. Note that the curve is solid at the point $(1, 7)$. Therefore, $\boldsymbol{h(1) = 7.}$

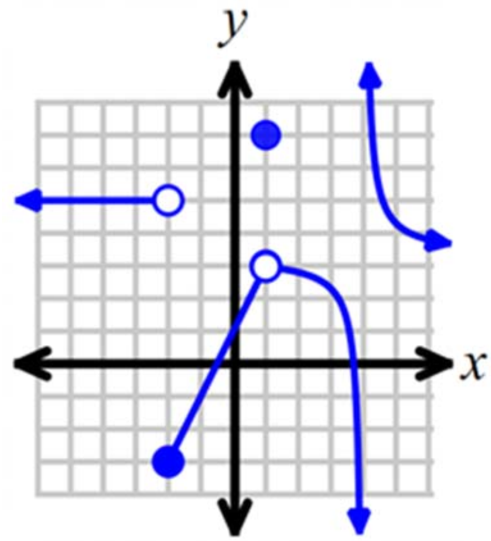
Answer A

4) $\lim_{x \to 1} h(x)$

A) 7 B) 1 (C) 3 D) DNE

The limits from the left and right are the same @ $x = 1$. From both the left and right, the curve approaches $y = 3$. The value of the function @ $x = 1$ is not $y = 3$, but that affects continuity, not the value of the limit. Therefore, $\boldsymbol{\lim_{x \to 1} h(x) = 3.}$

Answer C



5) $\lim_{x\to 1^+} h(x)$

A) 7 B) 1 C) 3 D) DNE

From the right, the curve approaches $y = 3$ as x approaches 1. The value of the function @ $x = 1$ is not $y = 3$, but that does not affect the value of the limit. Therefore, $\mathbf{\lim_{x\to 1^+} h(x) = 3.}$

Answer C

6) $\lim_{x\to 1^-} h(x)$

A) 7 B) 1 C) 3 D) DNE

From the left, the curve approaches $y = 3$ as x approaches 1. The value of the function @ $x = 1$ is not $y = 3$, but that does not affect the value of the limit. Therefore, $\mathbf{\lim_{x\to 1^-} h(x) = 3.}$

Answer C

7) $h(4)$

A) 8 B) 0 C) -1 D) DNE

This question asks for the value of the function @ $x = 4$. We can see @ $x = 4$ that the function is asymptotic to the vertical line $x = 4$. Therefore, **the function does not have a value at $x = 4$.**

Answer D

A discontinuity at a vertical asymptote is a type of "essential discontinuity" called an "infinite discontinuity." See page 9 in the Calculus Handbook for more information on types of discontinuities.

8) $\lim_{x\to 4} h(x)$

A) 8 B) 0 C) -1 D) DNE

The limits from the left and right are different @ $x = 4$. From the left, the curve approaches $y = -\infty$, and from the right, the curve approaches $y = +\infty$. Therefore, **the limit does not exist.**

Answer D

9) $\lim_{x\to -3} h(x)$

A) 5 B) 0 C) -5 D) DNE

The limits from the left and right are the same @ $x = -3$. Notice that the curve is continuous @ $x = -3$; there is no weirdness going on there. If a curve is continuous at a point, the limit exists and is equal to the value of the function at that point.

At $x = -3$, from both the left and right, the curve approaches $y = 5$. Therefore, $\mathbf{\lim_{x\to -3} h(x) = 5.}$

Answer A

10) Identify any x-values where $h(x)$ has a discontinuity but the limit exists.

This question asks for discontinuities where the limit exists. Of the discontinuities shown in the graph, namely at $x = -2, 1, 4$, the only location where the limit exists is @ $\mathbf{x = 1.}$

For #11 – 17, evaluate each limit. Simplify your answer.

Techniques for finding the values of limits are discussed on pages 12-15 of the Calculus Handbook. Generally, the approaches used by students should be, in order: substitution, simplification, rationalization, and finally, L'Hospital's Rule. L'Hospital's Rule is not part of the Precalculus curriculum and will not be used in this review.

11) $\lim_{x\to 10} \sqrt{5x-2}$

Substitution works fine for this limit. It is straightforward and contains no division. The student should always be on the lookout for division by zero in a limit.

$$\lim_{x\to 10} \sqrt{5x-2} = \sqrt{5(10)-2} = \sqrt{48} = \sqrt{16\cdot 3} = \mathbf{4\sqrt{3}}$$

12) $\lim_{x\to 2}(-3e^{x-2}-4)$

Substitution works fine for this limit. It is straightforward and contains no division.

$$\lim_{x\to 2}(-3e^{x-2}-4) = -3e^{2-2}-4 = -3e^{0}-4 = -3-4 = \mathbf{-7}$$

13) $\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16}$

Substitution will not work on this one because when $x = 16$, the expression in the limit becomes $\frac{0}{0}$. This is called an indeterminate form and requires techniques other than substitution. Either simplification or rationalizing the numerator will work on this limit.

Technique 1: Simplification

$$\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16} = \lim_{x\to 16} \frac{\sqrt{x}-4}{(\sqrt{x}-4)(\sqrt{x}+4)} = \lim_{x\to 16} \frac{1}{(\sqrt{x}+4)} = \frac{1}{(\sqrt{16}+4)} = \mathbf{\frac{1}{8}}$$

Technique 2: Rationalize the numerator (i.e., multiply by the conjugate of the numerator over itself)

$$\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16} = \lim_{x\to 16} \frac{\sqrt{x}-4}{x-16}\cdot\frac{(\sqrt{x}+4)}{(\sqrt{x}+4)} = \lim_{x\to 16} \frac{(x-16)}{(x-16)(\sqrt{x}+4)} = \lim_{x\to 16} \frac{1}{(\sqrt{x}+4)} = \frac{1}{(\sqrt{16}+4)} = \mathbf{\frac{1}{8}}$$

14) $\lim_{x\to -3} \frac{2x^2+6x}{x^2-9}$

Substitution will not work on this one because when $x = -3$, the expression in the limit becomes $\frac{0}{0}$. Let's try simplification.

$$\lim_{x\to -3} \frac{2x^2+6x}{x^2-9} = \lim_{x\to -3} \frac{2x(x+3)}{(x-3)(x+3)} = \lim_{x\to -3} \frac{2x}{(x-3)} = \frac{2(-3)}{(-3-3)} = \mathbf{1}$$

15) $\lim_{x\to 0} \frac{\sqrt{x+64}-8}{x}$

Substitution will not work on this one because when $x = 0$, the expression in the limit becomes $\frac{0}{0}$. The limit also looks very difficult to simplify. Let's try rationalizing the numerator.

$$\lim_{x\to 0} \frac{\sqrt{x+64}-8}{x} = \lim_{x\to 0} \frac{\sqrt{x+64}-8}{x} \cdot \frac{\left(\sqrt{x+64}+8\right)}{\left(\sqrt{x+64}+8\right)} = \lim_{x\to 0} \frac{x+64-64}{x \cdot \left(\sqrt{x+64}+8\right)}$$

$$= \lim_{x\to 0} \frac{x}{x \cdot \left(\sqrt{x+64}+8\right)} = \lim_{x\to 0} \frac{1}{\left(\sqrt{x+64}+8\right)} = \frac{1}{\left(\sqrt{0+64}+8\right)} = \mathbf{\frac{1}{16}}$$

16) $\lim_{x\to -10} \sqrt[3]{x+2}$

Substitution works fine for this limit. It is straightforward and contains no division.

$$\lim_{x\to -10} \sqrt[3]{x+2} = \sqrt[3]{-10+2} = \sqrt[3]{-8} = \mathbf{-2}$$

For #17 – 21, use $f(x) = \begin{cases} -4e^{x^2-9}+5 \text{ if } x < 3 \\ 8x-20 \text{ if } x = 3 \\ 4-x \text{ if } x > 3 \end{cases}$

17) Find $\lim_{x\to 3^-} f(x)$.

Since we are only concerned with what happens to the left of $x = 3$, we only need to worry about the function to the left of $x \to 3$, i.e., $f(x) = -4e^{x^2-9}+5$.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \left(-4e^{x^2-9}+5\right) = -4e^{3^2-9}+5 = -4+5 = \mathbf{1}$$

18) Find $\lim_{x\to 3^+} f(x)$.

Since we are only concerned with what happens to the right of $x = 3$, we only need to worry about the function to the right of $x \to 3$, i.e., $f(x) = 4 - x$.

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (4-x) = 4-3 = \mathbf{1}$$

19) Find $\lim_{x\to 3} f(x)$.

$\lim_{x\to 3} f(x)$ exists if $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = L$, and in that case, $\lim_{x\to 3} f(x) = L$. Since the limits from both the left and right are equal (see above), they give us the value of the limit. That is: $\mathbf{\lim_{x\to 3} f(x) = 1}$

20) Find $f(3)$.

Find this value directly from the definition of the function @ $x = 3$:

$$f(x) = 8x - 20 \quad \Rightarrow \quad f(3) = 8(3) - 20 = \mathbf{4}$$

Continuity: A function, f, is continuous at $x = a$ iff:

- $f(a)$ is defined,
- $\lim\limits_{x\to a} f(x)$ exists, and
- $\lim\limits_{x\to a} f(x) = f(a)$
- If $x = a$ is an endpoint, then the limit need only exist from the left or the right.

Note: $\lim\limits_{x\to a} f(x)$ exists if and only if:
$\lim\limits_{x\to a-} f(x) = \lim\limits_{x\to a+} f(x)$.

21) Is $f(x)$ continuous at $x = 3$? Use the definition of continuity.

Using the results of above problems we see that the limit exists, but is not equal to the value of $f(x)$ at $x = 3$. Therefore, $\boldsymbol{f(x)}$ **is not continuous at** $\boldsymbol{x = 3}$**.**

For #22 – 24, find $f'(x)$ by using the }µ- u#kmru¡ u#£¡ of a derivative. Show your work (do not use power rule except to verify your result.)

22) $f(x) = 5x - 1$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{[5(x+h) - 1] - [5x - 1]}{h}$$

$$= \lim_{h\to 0}\frac{[5x + 5h - 1] - [5x - 1]}{h} = \lim_{h\to 0}\frac{5h}{h} = \mathbf{5}$$

Notice that all of the terms in $f(x)$ cancel on the left and right sides of the numerator, and all that is left in the numerator is one or more terms involving h.

23) $f(x) = x^2 + 2x - 1$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{[(x+h)^2 + 2(x+h) - 1] - [x^2 + 2x - 1]}{h}$$

$$= \lim_{h\to 0}\frac{[x^2 + 2hx + h^2 + 2x + 2h - 1] - [x^2 + 2x - 1]}{h}$$

$$= \lim_{h\to 0}\frac{2hx + h^2 + 2h}{h} = \lim_{h\to 0}(2x + h + 2) = \mathbf{2x + 2}$$

24) $f(x) = -3\sqrt{x}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{[-3\sqrt{x+h}] - [-3\sqrt{x}]}{h} = -3\lim_{h\to 0}\frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= -3\lim_{h\to 0}\left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\cdot\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right) = -3\lim_{h\to 0}\left(\frac{x + h - x}{h\cdot(\sqrt{x+h} + \sqrt{x})}\right)$$

$$= -3\lim_{h\to 0}\left(\frac{h}{h\cdot(\sqrt{x+h} + \sqrt{x})}\right) = -3\lim_{h\to 0}\frac{1}{(\sqrt{x+h} + \sqrt{x})} = -3\left(\frac{1}{\sqrt{x} + \sqrt{x}}\right) = \frac{\mathbf{-3}}{\mathbf{2\sqrt{x}}}$$

For #25 – 26: Given that $h(x) = -2x^3 + 3x^2 + 4$

25) Find the slope of the tangent line to $h(x)$ at $(-2, 32)$.

The slope of the tangent line at a point is equal to the value of the derivative at that point. In Calculus notation,

$$h'(-2) = \frac{d}{dx}(-2x^3 + 3x^2 + 4)\Big|_{x=-2}$$

Using the power rule,

$$h'(-2) = (-6x^2 + 6x)\Big|_{x=-2} = -6(-2)^2 + 6(-2) = \mathbf{-36}$$

26) Write the equation of the tangent line to $h(x)$ in (h, k) form at $(-2, 32)$.

In order to define a tangent line, we need a slope and a point. From the problem above, we have a slope of -36. A point on the line is given to be $(-2, 32)$.

An equation of the tangent line in (h, k) form, then, is:

$$y = -36(x - (-2)) + 32 \quad \Rightarrow \quad \boldsymbol{y = -36(x + 2) + 32}$$

For #27 – 37, find the derivative of each function.

27) $g(x) = -5x^2 + 10x - 3$

Using the power rule, $\boldsymbol{g'(x) = -10x + 10}$

28) $y = 5\sqrt{x} - \frac{1}{x^3} + \pi$

Rewrite the equation in terms of powers of x, then use the power rule.

$$y = 5x^{1/2} - x^{-3} + \pi$$

$$\boldsymbol{y' = \frac{5}{2}x^{-1/2} + 3x^{-4} = \frac{5}{2\sqrt{x}} + \frac{3}{x^4}}$$

29) $f(x) = \dfrac{8x^3 - 4x^2 + 5x}{2x}$

Simplify the equation, then use the power rule.

$$f(x) = \frac{8x^3 - 4x^2 + 5x}{2x} = 4x^2 - 2x + \frac{5}{2}$$

$$\boldsymbol{f'(x) = 8x - 2}$$

30) $y = (2x-3)(5x+4)$

Method 1: Expand the equation, then use the power rule.

$y = (2x-3)(5x+4) = 10x^2 - 7x - 12$

$\boldsymbol{y' = 20x - 7}$

Method 2: Use the product rule.

$y = (2x-3)(5x+4)$

$y' = (2x-3)\cdot 5 + (5x+4)\cdot 2 = 10x - 15 + 10x + 8 = \boldsymbol{20x - 7}$

31) $h(x) = -3\sqrt{x}(4x^2+4x)$

Multiply the terms through, then use the power rule.

$h'(x) = -3\sqrt{x}(4x^2+4x) = -12x^{5/2} - 12x^{3/2} = \boldsymbol{-30x^{3/2} - 18x^{1/2}}$

32) $y = (2x^{-3}+5x)(x^2-4x^{-2})$

Yuk! Expand the equation, then use the power rule. Using the product rule would be a doozy!

$y = (2x^{-3}+5x)(x^2-4x^{-2}) = 2x^{-1} - 8x^{-5} + 5x^3 - 20x^{-1}$

$= 5x^3 - 18x^{-1} - 8x^{-5}$

$y' = \boldsymbol{15x^2 + 18x^{-2} + 40x^{-6}}$

33) $f(x) = \frac{7x-3}{x+4}$

No easy way around using the quotient rule for this one.

$f(x) = \frac{7x-3}{x+4}$

$$f'(x) = \frac{(x+4)\cdot 7 - (7x-3)\cdot 1}{(x+4)^2} = \frac{7x+28-7x+3}{(x+4)^2} = \boldsymbol{\frac{31}{(x+4)^2}}$$

34) $y = -\frac{5}{x^2+3}$

Rewrite the equation using a negative exponent, then use the power and chain rules.

$y = -\frac{5}{x^2+3} = -5(x^2+3)^{-1}$

$y' = -5\cdot -(x^2+3)^{-2}\cdot 2x$ chain rule

$= \boldsymbol{10x(x^2+3)^{-2} = \frac{10x}{(x^2+3)^2}}$

35) $g(x) = \frac{x^2+5x}{3x-1}$

No easy way around using the quotient rule for this one.

$$g(x) = \frac{x^2 + 5x}{3x - 1}$$

$$g'(x) = \frac{(3x-1)(2x+5) - (x^2+5x)(3)}{(3x-1)^2}$$

$$= \frac{(6x^2+13x-5) - (3x^2+15x)}{(3x-1)^2} = \frac{\mathbf{3x^2 - 2x - 5}}{\mathbf{(3x-1)^2}} = \frac{\mathbf{(3x-5)(x+1)}}{\mathbf{(3x-1)^2}}$$

36) $h(x) = -2(3x+11)^4$

Use the power and chain rules.

$$h(x) = -2(3x+11)^4$$

$$h'(x) = -8(3x+11)^3 \cdot 3 \qquad \text{chain rule}$$

$$= \mathbf{-24(3x+11)^3}$$

37) $m(x) = -5\sqrt{6x-8}$

Rewrite the equation using a fractional exponent, then use the power and chain rules.

$$m(x) = -5\sqrt{6x-8} = -5(6x-8)^{1/2}$$

$$m'(x) = -\frac{5}{2}(6x-8)^{-1/2} \cdot 6 \qquad \text{chain rule}$$

$$= \mathbf{-15(6x-8)^{-1/2}} = \frac{\mathbf{-15}}{\mathbf{\sqrt{6x-8}}}$$

38) $y = -5(1-8x)^{-3}$

Use the power and chain rules.

$$y = -5(1-8x)^{-3}$$

$$y' = 15(1-8x)^{-4} \cdot (-8) \qquad \text{chain rule}$$

$$= \mathbf{-120(1-8x)^{-4}}$$

For #39 – 41: Multiple Choice. Use the table shown to find the requested derivative.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
4	−9	3	5	−3
−1	8	−1	−2	6
6	1	−5	4	−4
−5	−10	7	20	−8

39) Find $h'(-1)$ if $h(x) = f(x) \cdot g(x)$.

A) -12 B) -8 C) 46 D) 50

$h(x)$ is the product of functions $f(x)$ and $g(x)$. Use the product rule.

$$h'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

$$h'(-1) = f(-1) \cdot g'(-1) + g(-1) \cdot f'(-1)$$

$$= 8 \cdot 6 + (-1)(-2) = 48 + 2 = \mathbf{50}$$

Answer D

40) Find $w'(4)$ if $w(x) = \dfrac{f(x)}{g(x)}$.

A) $-\frac{4}{3}$ B) -4 C) -1 D) $-\frac{5}{3}$

$w(x)$ is the quotient of functions $f(x)$ and $g(x)$. Use the quotient rule.

$$w'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$$

$$w'(4) = \frac{g(4) \cdot f'(4) - f(4) \cdot g'(4)}{\left(g(4)\right)^2}$$

$$= \frac{3 \cdot 5 - (-9)(-3)}{3^2} = -\frac{12}{9} = -\mathbf{\frac{4}{3}}$$

Answer A

41) Find $k'(6)$ if $k(x) = f\left(g(x)\right)$.

A) -12 B) -20 C) -80 D) -160

$k(x)$ is a composition of functions $f(x)$ and $g(x)$. Use the chain rule.

$$k'(x) = f'(g(x)) \cdot g'(x)$$

$$k'(6) = f'(g(6)) \cdot g'(6)$$

$$= [f'(-5)] \cdot (-4)$$

$$= 20 \cdot (-4) = \mathbf{-80}$$

Answer C