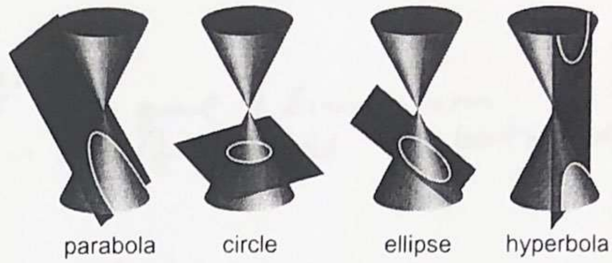
9.0 Notes: Introduction to Conics



Type of Conic and Standard Form	Key Features and Basic Hints (We will learn more details about graphing conics this unit)
Circle $(x-h)^2+(y-k)^2=r^2$	Center = (h,k) Radius = r The circle is r units from the center in all directions
Ellipse $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$	Center = (h,k) Axes length = $2a$ and $2b$ Major axis: the longer axis Minor axis: the shorter axis The endpoints of the horizontal axis are $\pm a$ units left and right from the center The endpoints of the vertical axis are $\pm b$ units up and down from the center
Hyperbola (opens horizontally) $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$	Center = (h,k) the transverse axis is parallel to the x-axis the coordinates of the vertices are $(h\pm a,k)$ the coordinates of the co-vertices are $(h,k\pm b)$ Draw a rectangle that includes the vertices and co-vertices; draw asymptotes as diagonals of the rectangle. Use the vertices and the asymptotes to draw the horizontal hyperbola.
Hyperbola (opens vertically) $\dfrac{(y-k)^2}{a^2}-\dfrac{(x-h)^2}{b^2}=1$	Center = (h,k) the transverse axis is parallel to the x-axis the coordinates of the vertices are $(h\pm a,k)$ the coordinates of the co-vertices are $(h,k\pm b)$ Draw a rectangle that includes the vertices and co-vertices; draw asymptotes as diagonals of the rectangle. Use the vertices and the asymptotes to draw the horizontal hyperbola.
Parabola: *vertex form* **(opens vertically)** $y=a(x-h)^2+k$	Vertex (h,k) Use $\frac{a}{1}$ as $\frac{rise}{run}$ to plot one point on either side of the vertex.
Parabola: *vertex form* **(opens horizontally)** $x=a(y-k)^2+h$	Vertex (h,k) Use $\frac{a}{1}$ as $\frac{run}{rise}$ to plot one point on either side of the vertex.

9.0 Notes, continued.

Completing the Square (Converting to Standard Form)

① factor out leading coeff from quad & linear term
② new linear term → b → use $\left(\frac{b}{2}\right)^2$ + add to both sides
③ factor

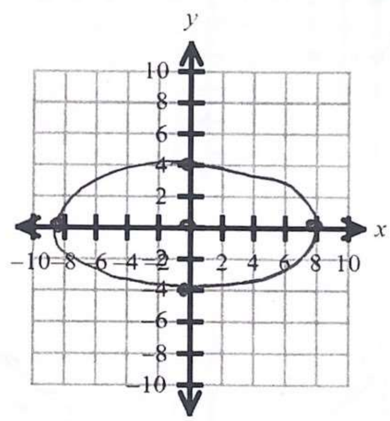
- Circles $ax^2 + ay^2$ same leading coeff → if needed, divide to make lead coeff = 1
- Ellipses and hyperbolas
  diff leading coeff $x^2 - y^2$ or $y^2 - x^2$ Both = 1 divide as needed
  +
- Parabolas $y = a(x-h)^2 + k$ } $x = a(y-k)^2 + h$
  one quadratic term & one linear term

For Examples 1 – 4, identify the type of conic section represented, convert to Standard Form, and then graph each conic.

1) $\frac{x^2}{64} + \frac{4y^2}{64} = \frac{64}{64}$ ← ellipse

$$\boxed{\frac{x^2}{64} + \frac{y^2}{16} = 1}$$

$a = 8$ ← → $b = 4$ ↑ ↓ center $(0,0)$

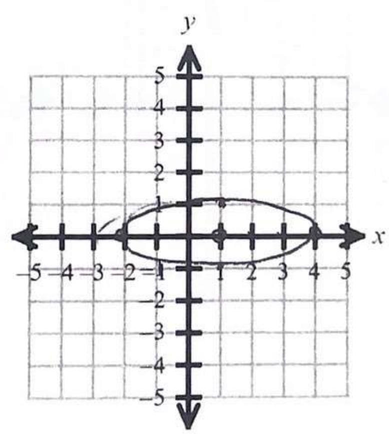


2) $x^2 + 9y^2 - 2x - 8 = 0$

$(x^2 - 2x + 1) + 9y^2 = 8 + 1$

$\frac{(x-1)^2}{9} + \frac{9y^2}{9} = \frac{9}{9}$ ellipse

$\frac{(x-1)^2}{9} + \frac{y^2}{1} = 1$ center $(1,0)$ ← 3 → ↑ 1 ↓



9.0 Notes, continued.

3) $\cancel{9x^2 - 4y^2 - 18x + 8y = 31}$ hyperbola

$16x^2 - 4y^2 - 64x + 40y = 100$

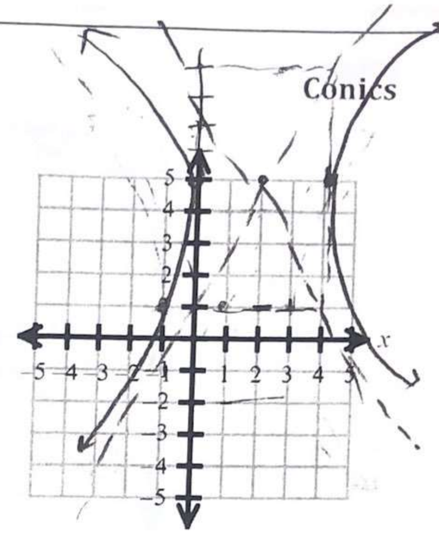
$16(x^2 - 4x + 4) - 4(y^2 - 10y + 25) = 100 + 64 - 100$

$16(x-2)^2 - 4(y-5)^2 = 64$

$$\frac{(x-2)^2}{4} - \frac{(y-5)^2}{16} = 1$$

center $(2,5)$

$\leftrightarrow 2$ $\updownarrow 4$



4) $y - x^2 + 8x = \cancel{6}\ 13$ parabola

$y = x^2 - 8x + 13$

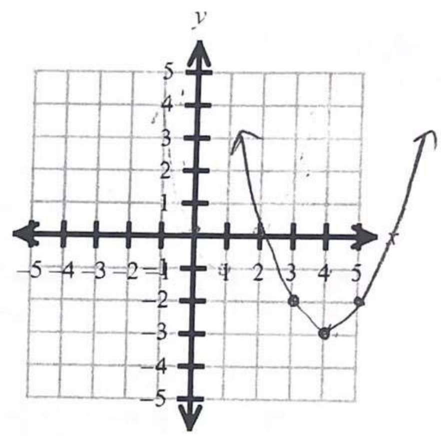
$y = (x^2 - 8x + \underline{16}) - \underline{16} + 13$

$\left(\frac{-8}{2}\right)^2$

$y = (x-4)^2 - 3$

vertex $(4,-3)$

$\frac{a}{1} = \frac{1 \uparrow}{1 \leftrightarrow}$



5) $25y^2 - 100y + 100 - x^2 = 25$ hyperbola

$25(y^2 - 4y + 4) - x^2 = 25 \cancel{+} 100 + 100$

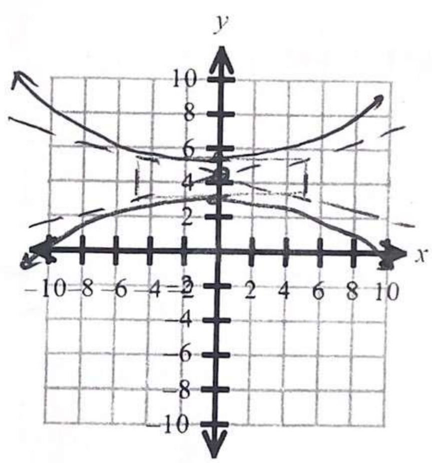
$25(y-2)^2 - x^2 = 25$

$$\frac{(y-2)^2}{1} - \frac{x^2}{25} = 1$$

center $(0,2)$

$\leftarrow 5 \rightarrow$

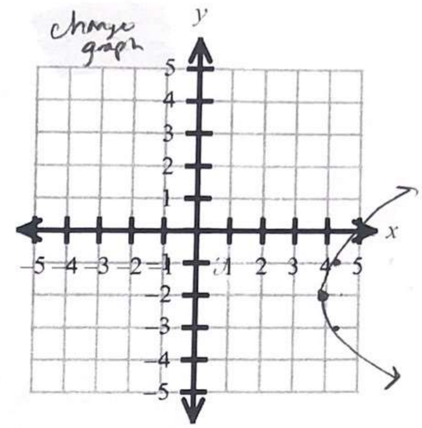
$\updownarrow$



9.0 Notes, continued.

6) $y^2 + 4y - 3x = -16$ parabola

change graph



$(y^2+4y+4) - 4 + 16 = 3x$

$(y+2)^2 + 12 = 3x$

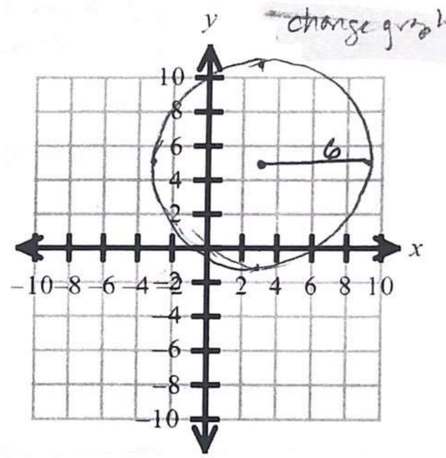
$x = \frac{1}{3}(y+2)^2 + 4$

vertex $(4, -2)$

$\frac{a}{1} \rightarrow \frac{\frac{1}{3}}{1}$ $\rightarrow$ $\updownarrow$

7) $x^2 - 6x - 10y = 2 - y^2$

change graph



$(x^2-6x+9) + (y^2-10y+25) = 2+25+9$

$(x-3)^2 + (y-5)^2 = 36$

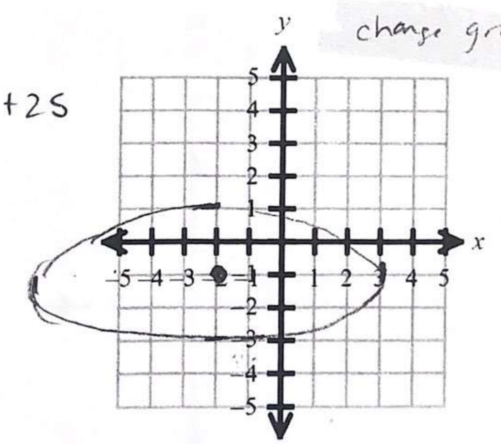
circle

center $(3, 5)$

$r = 6$

8) $4x^2 + 25y^2 + 16x + 50y - 59 = 0$ ellipse

change graph



$4(x^2+4x+4) + 25(y^2+2y+1) = 59+16+25$

$4(x+2)^2 + 25(y+1)^2 = 100$

$\frac{(x+2)^2}{25} + \frac{(y+1)^2}{4} = 1$

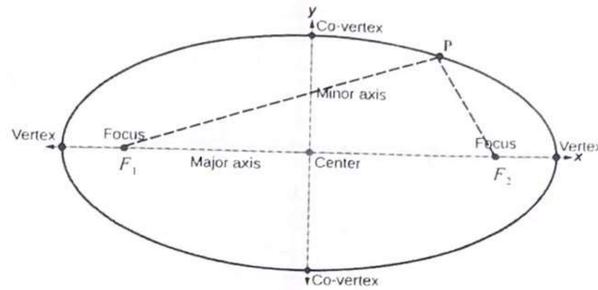
center $(-2, -1)$

$\leftarrow 5 \rightarrow$

$\updownarrow 4$

An ellipse is the set of all points P in a plane such that the sum of the distances from two fixed points F_1 and F_2 is constant.

- These two fixed points are called the foci.
- The midpoint of the segment connecting the foci is the center of the ellipse.



Standard Form of an Ellipse

Centered at the origin	Centered at (h, k)
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **Center:** $(0,0)$ $\leftarrow a \rightarrow$ $\updownarrow b$	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ **Center:** (h,k) $\leftarrow a \rightarrow$ $\updownarrow b$

*If $a^2 > b^2$, then the $\underbrace{\text{major axis}}_{\text{longer axis}}$ is horizontal.

*If $b^2 > a^2$, then the $\underbrace{\text{major axis}}_{\text{longer axis}}$ is vertical.

Foci of an Ellipse

Location	Foci Formula
• on the major axis • c units from the center in 2 directions	$b^2 = a^2 - c^2$ or $c^2 = a^2 - b^2$

Example 1: Graph the following and locate the foci:

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

$a=3 \quad b=2$

$c^2 = 9 - 4$

$c^2 = 5$

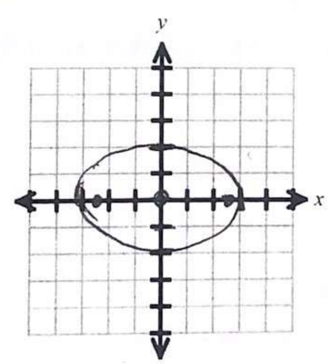
$c = \pm\sqrt{5}$

center $(0,0)$

$\leftarrow 3 \rightarrow$ major

$\uparrow 2 \downarrow$

foci $\rightarrow$ $\boxed{(\sqrt{5},0) \quad (-\sqrt{5},0)}$



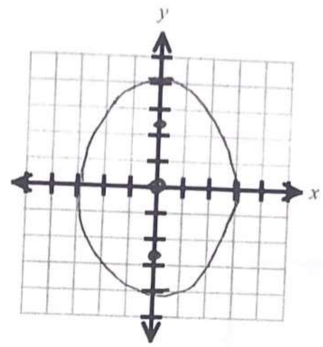
Example 2: Graph the following and locate the foci:
$16x^2 + 9y^2 = 144$

$\frac{x^2}{9} + \frac{y^2}{16} = 1$

center $(0,0)$
$\leftarrow 3 \rightarrow$
$\uparrow$ 4 major $\downarrow$

$c^2 = 16 - 9$
$c^2 = 7$
$c = \pm\sqrt{7}$

foci: $(0, \sqrt{7})$ $(0, -\sqrt{7})$



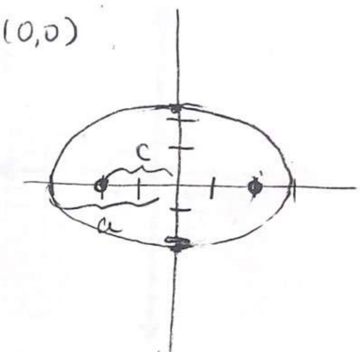
For #3 – 5, find the standard form of the equation of each ellipse with the given information.

3) foci at $\underbrace{(-2, 0) \;\&\; (2, 0)}_{\text{along major axis}}$ and vertices at (-3,0) and (3, 0)

$c = 2$ center $(0,0)$
$a = 3$

$4 = 9 - b^2$
$b^2 = 5$
$b = \pm\sqrt{5}$

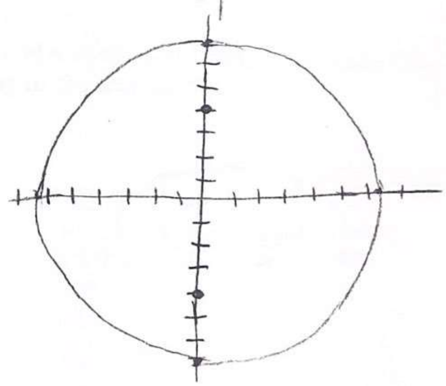
$$\boxed{\frac{x^2}{9} + \frac{y^2}{5} = 1}$$



4) foci at (0, -4) & (0, 4) and vertices at (0, -7) and (0, 7)

$c = 4$ $a = 7$
$16 = 49 - b^2$
$b^2 = 33$
$b = \sqrt{33}$

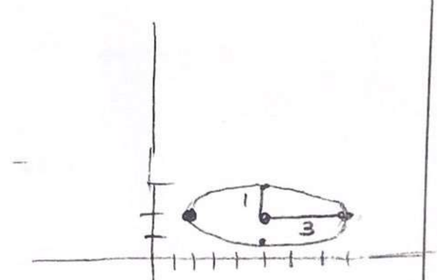
$$\boxed{\frac{x^2}{33} + \frac{y^2}{49} = 1}$$



5) endpoints of the major axis at (2, 2) and (8, 2); endpoints of the minor axis at (5, 3) and (5, 1)

center
$(5, 2)$

$$\boxed{\frac{(x-5)^2}{9} + \frac{(y-2)^2}{1} = 1}$$



For #6 – 7: Graph each ellipse and locate the foci.

6) $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$

center $(-1,2)$

$\leftarrow \overset{a}{3} \rightarrow$ major

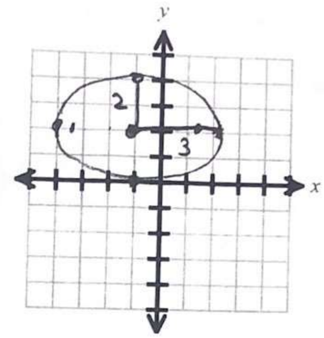
$\uparrow 2\ b \downarrow$

$c^2 = 9 - 4$

$c^2 = 5$

$c = \pm\sqrt{5}$ from center $\leftarrow \rightarrow$

$\boxed{\text{foci: } (-1+\sqrt{5}, 2) \text{ and } (-1-\sqrt{5}, 2)}$



7) $9x^2 + 25y^2 - 36x + 50y - 164 = 0$

$9(x^2 - 4x + 4) + 25(y^2 + 2y + 1) = 164 + 36 + 25$

$9(x-2)^2 + 25(y+1)^2 = 225$

$\frac{(x-2)^2}{25} + \frac{(y+1)^2}{9} = 1$

center $(2,-1)$

$\leftarrow 5 \rightarrow$ major

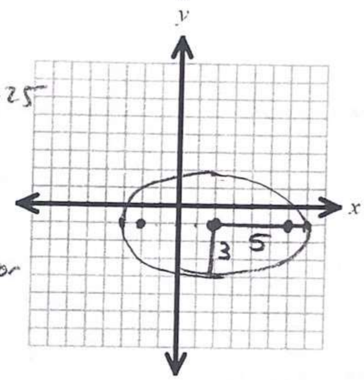
$\uparrow 3$ minor $\downarrow$

$c^2 = 25 - 9$

$c^2 = 16$

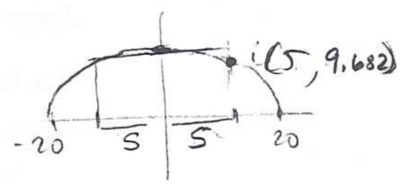
$c = \pm 4$ from center

$\boxed{\text{foci: } (6,-1) \text{ and } (-2,-1)}$



8) A semi-elliptical archway over a one-way road has a height of 10 feet and a width of 40 feet. Your truck has a width of 10 feet and a height of 9 feet. Will your truck clear the opening of the archway?

$\frac{x^2}{400} + \frac{y^2}{100} = 1$



$x^2 + 4y^2 = 400$ at $x = 10$

$25 + 4y^2 = 400$

$4y^2 = 375$

$y^2 = 93.75 \rightarrow y = 9.682$ ft

yes, room for the truck

Will a truck that is 12 ft wide and has a height of 9ft clear the opening?

$\leftarrow 6 \cdot 6 \rightarrow$

$x^2 + 4y^2 = 400$

$36 + 4y^2 = 400$

$4y^2 = 364$

$y^2 = 91 \rightarrow y = 9.539$ ft

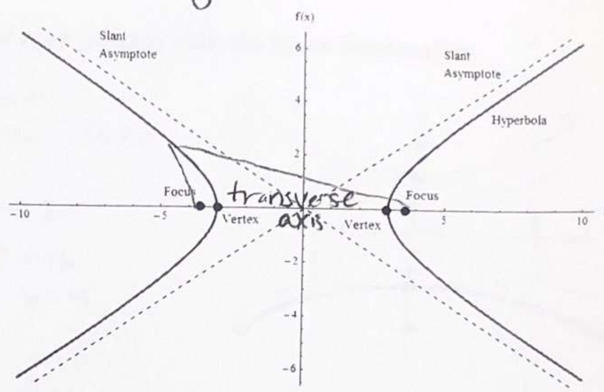
Yes, still has room

An exploration with ellipses: https://www.geogebra.org/m/CrVpc8pP

9.2 Notes: Hyperbolas

A hyperbola is the set of all points in a plane whose *difference* from two fixed points, called foci, is constant.

- **Vertices:** Two points that are intersected by a line segment that joins the foci
- **Transverse axis:** The line segment that joins the vertices



Standard Form of a Hyperbola

	Centered at the origin	Centered at (h,k)	Transverse Axis	Asymptotes
Opens horizontally	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ Center: $(0,0)$	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ Center: (h,K)	$\longleftrightarrow$ through center	slope is $\pm\frac{b}{a}$ through center
Opens vertically	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ Center: $(0,0)$	$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ Center: (h,K)	$\updownarrow$ through center	slope is $\pm\frac{b}{a}$ through center

Foci of a Hyperbola

Location	Foci Formula
• on the transverse axis • c units from the center in two directions	$c^2 = a^2 + b^2$

Examples 1 – 2: Find the vertices and locate the foci:

1) $\frac{x^2}{25}-\frac{y^2}{16}=1$

center $(0,0)$

$\leftarrow 5 \rightarrow$ vertices $(5,0)$ & $(-5,0)$

$c^2=25+16$

$c^2=41$

$c=\pm\sqrt{41}$

foci $(\sqrt{41},0)$ and $(-\sqrt{41},0)$

2) $\frac{y^2}{25}-\frac{x^2}{16}=1$

center $(0,0)$

$\updownarrow$ transverse

Vertices $(0,5)$ and $(0,-5)$

$c^2=25+16$

$c^2=41$

$c=\pm\sqrt{41}$

foci $(0,\sqrt{41})$ and $(0,-\sqrt{41})$

For #3 – 5, write the standard form of the equation of each parable with the given information.

3) foci at (0, -5) and (0, 5) and vertices (0, -3) and (0, 3)

$c = 5$

center $(0,0)$

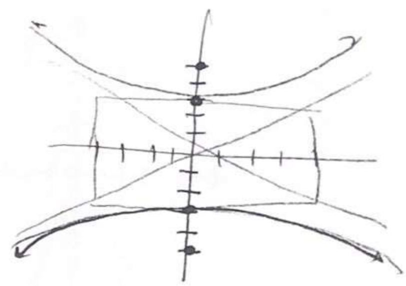
$a = 3$

$9 + b^2 = 25$

$b^2 = 16$

$b = 4$

$$\boxed{\frac{y^2}{9} - \frac{x^2}{16} = 1}$$



4) foci at (-4, 0) and (4, 0) and vertices (-3, 0) and (3, 0)

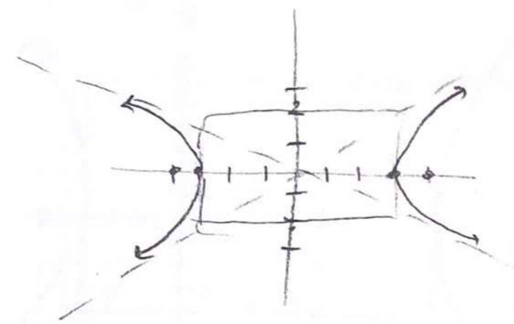
center $(0,0)$

$c = 4$ $a = 3$

$16 = 9 + b^2$

$7 = b^2$

$$\boxed{\frac{x^2}{9} - \frac{y^2}{7} = 1}$$



5) center at (4, -2), focus at (7, -2) and vertex at (6, -2)

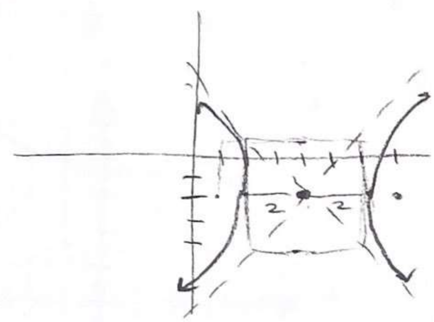
$c = 3$ $a = 2$

$9 = 4 + b^2$

$5 = b^2$

$\pm\sqrt{5} = b$

$$\boxed{\frac{x^2}{4} - \frac{y^2}{5} = 1}$$



6) Graph and locate the foci of the hyperbola below. Also write the equations of the asymptotes.

$$\frac{x^2}{36} - \frac{y^2}{9} = 1$$

$(0,0)$

$\leftarrow 6 \rightarrow$

$\uparrow 3 \downarrow$

$c^2 = 36 + 9$

$c^2 = 45$

$c = \pm 3\sqrt{5}$

foci $(3\sqrt{5}, 0)$ $(-3\sqrt{5}, 0)$

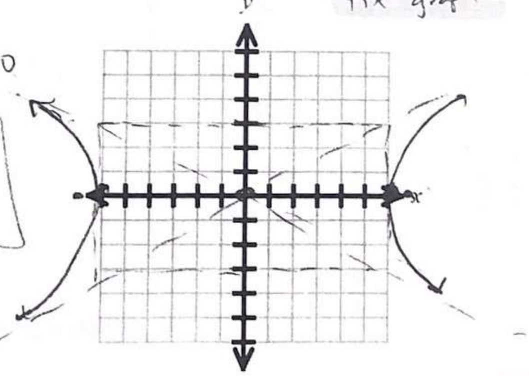
asymptotes

$y = mx + b$

$y = \pm\frac{3}{6}x + 0$

$$\boxed{y = \frac{1}{2}x \quad y = -\frac{1}{2}x}$$

fix graph



For #7 – 9, graph each hyperbola and locate the foci. Also, write the equations of the asymptotes.

7) $y^2 - 4x^2 = 4$

$\frac{y^2}{4} - \frac{x^2}{1} = 1$

$(0, 0)$

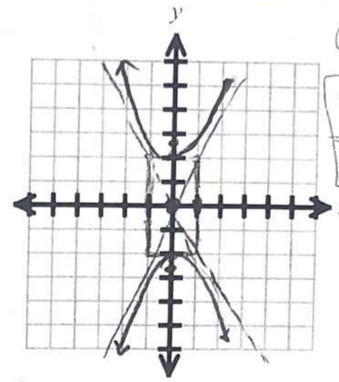
$\updownarrow 2 \quad \leftrightarrow 1$

$c^2 = 4 + 1$

$c^2 = 5$

$c = \pm\sqrt{5}$

Foci	$(0, \sqrt{5})$
	$(0, -\sqrt{5})$



Asymptotes

$\boxed{y = 2x}$

$\boxed{y = -2x}$

8) $\frac{(x-3)^2}{49} - \frac{(y-1)^2}{25} = 1$

center $(3, 1)$

$\leftrightarrow 7 \quad \updownarrow 5$

$c^2 = 49 + 25$

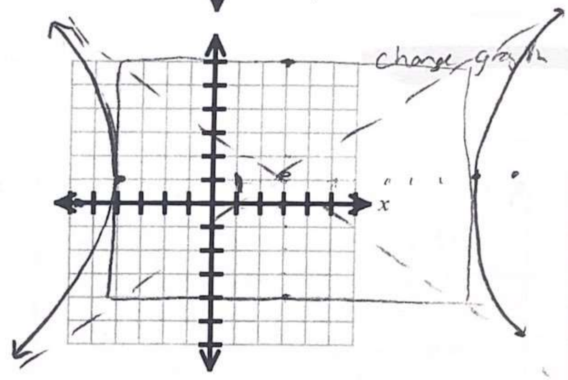
$c^2 = 74$

$c = \pm\sqrt{74}$

foci: $(3+\sqrt{74}, 1)$
$(3-\sqrt{74}, 1)$

asymptotes $y = m(x-h)+k$

$\boxed{y = \pm\frac{5}{7}(x-3)+1}$



change graph

9) $4x^2 - 24x - 25y^2 + 250y - 489 = 0$

$4(x^2 - 6x + 9) - 25(y^2 - 10y + 25) = 489 + 36 - 625$

$4(x-3)^2 - 25(y-5)^2 = -100$

$\frac{(y-5)}{4} - \frac{(x-3)^2}{25} = 1$

$(3, 5)$

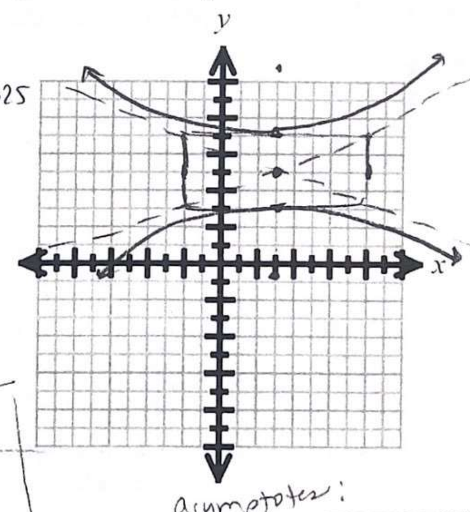
$\leftrightarrow 5 \quad \updownarrow 2$

$c^2 = 4 + 25$

$c^2 = 29$

$c = \pm\sqrt{29}$

foci $(3, 5+\sqrt{29})$
$(3, 5-\sqrt{29})$



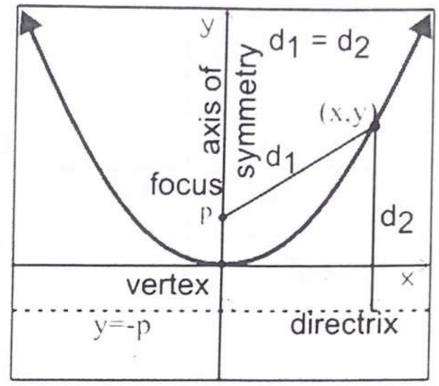
asymptotes:

$\boxed{y = \pm\frac{2}{5}(x-3)+5}$

Visualization of Hyperbolas and Foci: https://www.geogebra.org/m/cD5ZyfbC

9.3 Notes: Parabolas

A *parabola* is the set of all points in a plane that are equidistant from a fixed line (the directrix) and a fixed point (the focus).



In previous classes, you have explored the vertex form of a parabola:

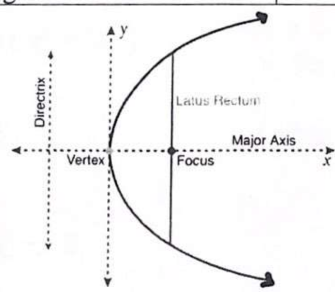
Parabola: *vertex form* **(opens vertically)**	**Parabola:** *vertex form* **(opens horizontally)**
$y = a(x-h)^2 + k$	$x = a(y-k)^2 + h$

Standard Form of a Parabola

	Centered at the origin	Centered at (h,k)	Focus	Directrix
Opens horizontally *left:* $p<0$ *right:* $p>0$	$y^2 = 4px$ **Center:** $(0,0)$	$(y-k)^2 = 4p(x-h)$ **Center:** (h,k)	p units away from vertex, inside curve, along axis of sym.	$-p$ units away from vertex, outside curve, ⊥ to axis of sym. $x=-p$
Opens vertically *up:* $p>0$ *down:* $p<0$	$x^2 = 4py$ **Center:** $(0,0)$	$(x-h)^2 = 4p(y-k)$ **Center:** (h,k)	''	$y=-p$ ''

Latus Rectum of a Parabola

Description	Length Formula
• Line segment that connects two points on the parabola • Is parallel to the directrix • Passes through the focus	The length of the latus rectum is $\lvert 4p \rvert$. This is helpful to determine the width of the parabola.



For Examples #1 – 4: Find the vertex, focus and directrix of the parabola and then graph. (use the latus directr

1) $y^2 = 8x$

$y^2 = 4px$

$8 = 4p$

$2 = p$

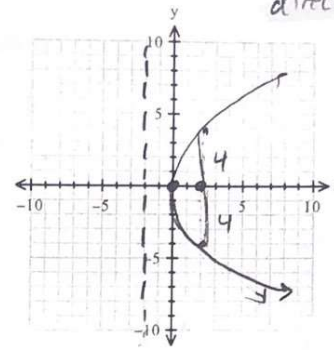
vertex $(0,0)$

focus $(2,0)$

directrix: $x = -2$

latus Rectum $= |4 \cdot 2|$

$= 8$



2) $x^2 = -12y$

$x^2 = 4py$

$-12 = 4p$

$-3 = p$

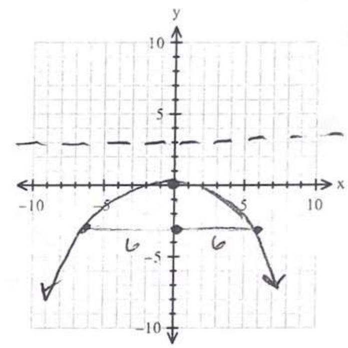
vertex $(0,0)$

focus $(0,-3)$

directrix: $y = 3$

latus rectum $= |4 \cdot 3| = 12$

$\leftarrow 6 \rightarrow$



3) $(x-2)^2 = 4(y+1)$

$(x-h)^2 = 4p(y-k)$

$4 = 4p$

$1 = p$

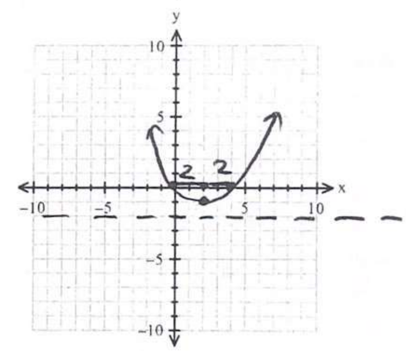
Vertex $(2,-1)$

focus $(2,0)$

directrix: $y = -2$

latus rectum $= |4| = 4$

$\leftarrow 2 \rightarrow$



4) $y^2 + 2y + 4x - 7 = 0$

$(y^2 + 2y + 1) = -4x + 7 + 1$

$(y+1)^2 = -4(x-2)$

$-4 = 4p$

$-1 = p$

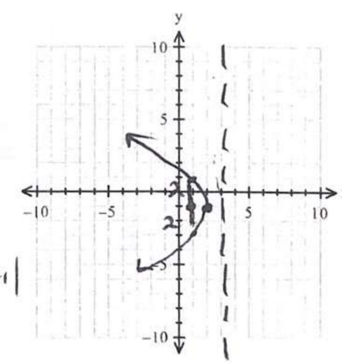
vertex $(2,-1)$

focus $(1,-1)$

directrix: $x = 3$

latus rectum $= |4 \cdot -1|$

$= 4$



9.3 Notes, continued.

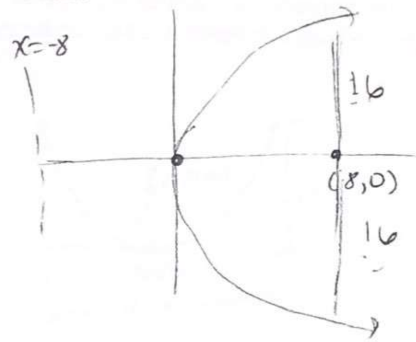
For #5 – 8, write each parabola in standard form with the given information.

5) focus (8, 0) and directrix $x = -8$

Vertex = (0, 0)

$p = 8$

$y^2 = 32x$

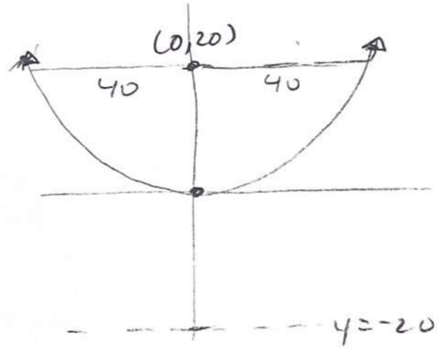


6) focus (0, 20) and directrix $y = -20$

Vertex = (0, 0)

$p = 20$

$\boxed{x^2 = 80y}$

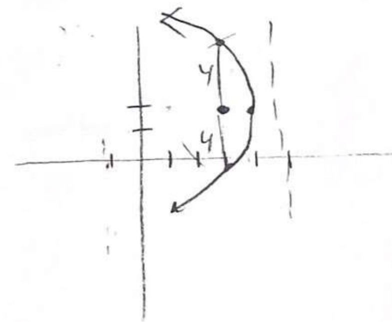


7) focus (3, 2) and directrix $x = 5$

vertex = (4, 2)

$p = -2$

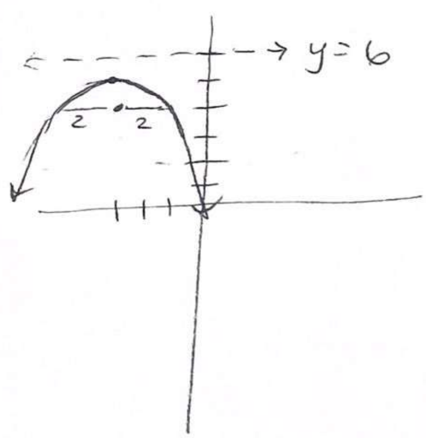
$(y-2)^2 = -8(x-4)$



8) focus (-3, 4) and directrix $y = 6$

vertex = (−3, 5) $p = -1$

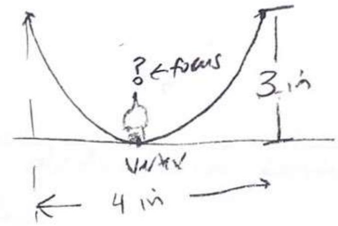
$(x+3)^2 = -4(y-5)$



9.3 Notes, continued.

opens up
assume the center is at (0,0)

9) An engineer is designing a flashlight using a parabolic reflecting mirror and a light source. The casting has a diameter of 4 inches and a depth of 3 inches. What is the equation of the parabola used to shape the mirror? At what point should the light source be placed relative to the mirror's vertex?



$x^2 = 4py$ focus $(0,p)$

Use $(2,3)$ as pt on parabola

$2^2 = 4 \cdot p \cdot 3$

$4 = 12p$

$\frac{1}{3} = p$

equation ⟹ $x^2 = 4 \cdot \frac{1}{3} y$

$$\boxed{x^2 = \frac{4}{3} y}$$

light placed at focus: $(0, \frac{1}{3})$ above vertex

9.5 Parametric Equations

*Khan Academy Video:
http://www.khanacademy.org/math/trigonometry/parametric_equations/parametric/v/parametric-equations-1

Key Terms

Parametric Equations	equations where x & y are defined in terms of a 3rd variable (usually t) $x = f(t)$ & $y = g(t)$
Parameter	the 3rd variable (usually t)
Plane Curve	The set of ordered pairs (x, y) where $x = f(t)$ and $y = g(t)$ for an interval I. *can be graphed on a coord. system

Examples:

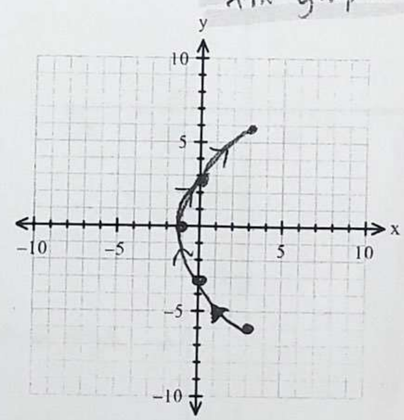
1) Graph the plane curve defined by the parametric equations below on the interval $-2 \le t \le 2$.

$x = t^2 - 1; y = 3t$

t	x (t^2-1)	y ($3t$)
-2	3	-6
-1	0	-3
0	-1	0
1	0	3
2	3	6

arrow follows t.

fix graph



9.5 Notes, continued.

2) Sketch the plane curve represented by the parametric equations $x = \sqrt{t}$ and $y = 2t - 1$ by eliminating the parameter.

$x^2 = t$ (note: $t \geq 0$)

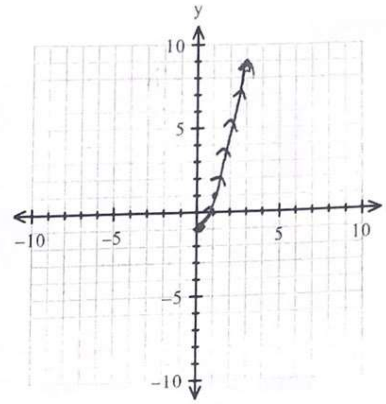
$y = 2(x^2) - 1$

$\boxed{y = 2x^2 - 1}$

① Isolate t in one equation

② Subst into other equation

③ Isolate y



3) Sketch the plane curve represented by the parametric equations below by eliminating the parameter.

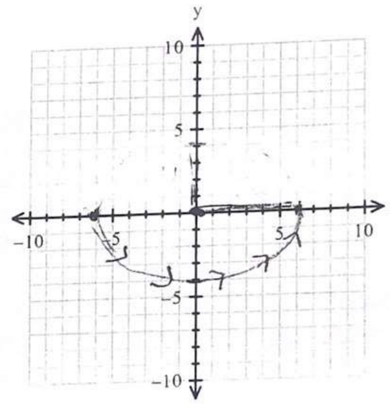
$x = 6\cos t\,;\, y = 4\sin t$ on the interval $\pi \leq t \leq 2\pi$

$\frac{x}{6} = \cos t \quad \frac{y}{4} = \sin t$

$\sin^2 t + \cos^2 t = 1$

$\frac{y^2}{16} + \frac{x^2}{36} = 1$

t	x	y
π	-6	0
$\frac{3\pi}{2}$	0	-4
2π	6	0

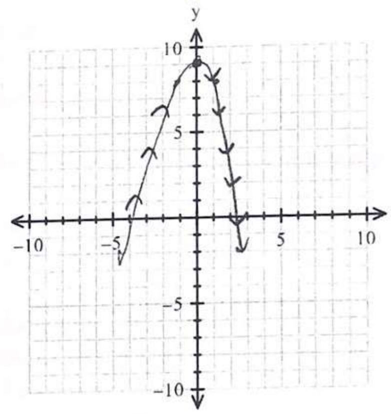


For #4 – 5: Find the set of parametric equations for each parabola.

4) $y = 9 - x^2$

$\boxed{\begin{array}{l} \text{Let } x = t \\ y = 9 - t^2 \end{array}}$

t	x	y
-2	-2	5
-1	-1	8
0	0	9
1	1	8
2	2	5

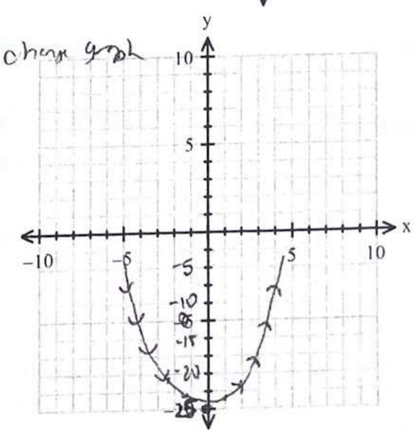


5) $y = x^2 - 25$

$\boxed{\begin{array}{l} \text{Let } x = t \\ y = t^2 - 25 \end{array}}$

t	x	y
-2	-2	-21
-1	-1	-24
0	0	-25
1	1	-24
2	2	-21

change graph



~~add practice problems!~~