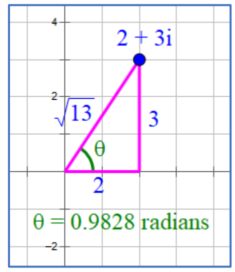
Complex Numbers

Forms:

- $a + bi$ $\quad$ $2 + 3i$
- $r(\cos\theta + i\sin\theta)$ $\quad$ $\sqrt{13}\,\text{cis}(0.9828)$
- $re^{i\theta}$ $\quad$ $\sqrt{13}\,e^{0.9828i}$



Note that:
$e^{i\theta} = \cos\theta + i\sin\theta$

See the Calculus Handbook
For two proofs of this.

The set of Complex Numbers ($\mathbb{C}$) are a field. So is the set of Real Numbers ($\mathbb{R}$).

- Closure law: $z_1 + z_2$ and $z_1 z_2$ are in $\mathbb{C}$.
- Commutative law of $+$: $z_1 + z_2 = z_2 + z_1$
- Associative law of $+$: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- Commutative law of $\cdot$: $z_1 \cdot z_2 = z_2 \cdot z_1$
- Associative law of $+$: $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$
- Distributive law: $z_1 \cdot (z_2 + z_3) = (z_1 \cdot z_2) + (z_1 \cdot z_3)$
- Additive Identity: $z_1 + 0 = z_1$
- Additive Inverse: $z_1 + (-z_1) = 0$
- Multiplicative Identity: $z_1 \cdot 1 = z_1$
- Multiplicative Inverse: $z_1 \cdot (1/z_1) = 1$

Implications of $e^{i\theta} = \cos\theta + i\sin\theta$

Use what you develop to answer each successive question. You may use your calculators:

1) What is: $e^{i\pi}$?

2) What is: $\ln(-1)$?

3) What is: $\ln(-37)$?

4) Recalling that $i = \sqrt{-1}$, what is i as a power of e, that is, $e^{what} = i$?

5) What is: $\ln(i)$?

6) What is: i^i? What is its approximate value (use a calculator)?

7) Using $e^{iz} = \cos z + i \sin z$ and $e^{-iz} = \cos z - i \sin z$, solve for $\cos z$.

8) Using $e^{iz} = \cos z + i \sin z$ and $e^{-iz} = \cos z - i \sin z$, solve for $\sin z$.

Answers:

1) $e^{i\pi} = \cos\pi + i\sin\pi = -1$
2) $\ln(-1) = \ln(e^{i\pi}) = i\pi$
3) $\ln(-37) = \ln(-1\cdot 37) = \ln(-1) + \ln(37) = i\pi + \ln(37)$. So, for any positive real number x, $\ln(-x) = i\pi + \ln x$.
4) $i = \sqrt{-1} = (-1)^{1/2} = (e^{i\pi})^{1/2} = e^{i\pi/2}$
5) $\ln(i) = \ln(e^{i\pi/2}) = i\pi/2$
6) $i^i = \left(e^{\frac{i\pi}{2}}\right)^i = e^{\frac{i^2\pi}{2}} = e^{-\frac{\pi}{2}} \sim 0.2079 \sim \frac{1}{5}$. Notice that i^i is a real number!
7)
$$\begin{array}{l} e^{iz} = \cos z + i\sin z \\ \underline{e^{-iz} = \cos z - i\sin z} \\ e^{iz} + e^{-iz} = 2\cos z \\ \dfrac{e^{iz} + e^{-iz}}{2} = \cos z \end{array}$$
8)
$$\begin{array}{l} e^{iz} = \cos z + i\sin z \\ \underline{-e^{-iz} = -\cos z + i\sin z} \\ e^{iz} - e^{-iz} = 2i\sin z \\ \dfrac{e^{iz} - e^{-iz}}{2i} = \sin z \end{array}$$