Complex Trigonometric Functions

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}$$

- Examples:

$$\cos i = \frac{e^{i \cdot i} + e^{-i \cdot i}}{2} = \frac{e^{-1} + e}{2} \sim 1.5431$$

$$\boxed{e^{i\pi} = -1, \quad e^{-i\pi} = \frac{1}{e^{i\pi}} = -1}$$

$$\tan(\pi - 2i) = \frac{\dfrac{\left(e^{i(\pi-2i)} - e^{-i(\pi-2i)}\right)}{2i}}{\dfrac{\left(e^{i(\pi-2i)} + e^{-i(\pi-2i)}\right)}{2}} = \frac{-i\left(e^{i\pi}e^{2} - e^{-i\pi}e^{-2}\right)}{\left(e^{i\pi}e^{2} + e^{-i\pi}e^{-2}\right)}$$

$$= -\frac{(e^{2} - e^{-2})}{(e^{2} + e^{-2})} i \sim -0.9640i$$

Complex Trigonometric Identities (most complex Trig identities match their real counterparts)

$$\sin(-z) = -\sin z$$

$$\cos(-z) = \cos z$$

$$\sin^2 z + \cos^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin(z_1)\cos(z_2) \pm \cos(z_1)\sin(z_2)$$

$$\cos(z_1 \pm z_2) = \cos(z_1)\cos(z_2) \mp \sin(z_1)\sin(z_2)$$

$$\sin(2z) = 2\sin(z)\cos(z)$$

$$\cos(2z) = \cos^2(z) - \sin^2(z)$$

Derivatives of Complex Trigonometric Functions

$$\frac{d}{dz}\sin z = \cos z \qquad \frac{d}{dz}\cos z = -\sin z$$

$$\frac{d}{dz}\tan z = \sec^2 z \qquad \frac{d}{dz}\cot z = -\csc^2 z$$

$$\frac{d}{dz}\sec z = \sec z \tan z \qquad \frac{d}{dz}\csc z = -\csc z \cot z$$

A Couple of Interesting Results Involving Complex Numbers

- ***i* to the power of *i*, i.e., i^i**

  $e^{i\theta} = \cos\theta + i\sin\theta$ (two proofs are in the Calculus Handbook)

  $e^{i\pi} = \cos\pi + i\sin\pi = -1$ This implies that: $e^{i\pi} + 1 = 0$

  $e^{i\pi/2} = \sqrt{-1} = i$

  $i = e^{i\pi/2}$

  $i^i = e^{i\cdot i\pi/2}$

  $i^i = e^{-\pi/2} \sim 0.20788$ (i^i is real and approximately 1/5)

- **Solving equations:** multiply equation by e^{iz} and use the quadratic equation. Example:

  $\sin z = 5,\ 0 \le \theta < 2\pi$ (weird, but possible in complex analysis)

  $\sin z = 5$

  $$\frac{e^{iz} - e^{-iz}}{2i} = 5$$

  $e^{iz} - 10i - e^{-iz} = 0$

  $e^{2iz} - 10ie^{iz} - 1 = 0$ note: this is quadratic in e^{iz}

  $$e^{iz} = \frac{10i \pm \sqrt{-100+4}}{2} = \left(5 \pm \sqrt{24}\right)i = \left(5 \pm 2\sqrt{6}\right)i$$

  $iz = \ln\left[\left(5 \pm 2\sqrt{6}\right)i\right]$

  $z = -i \cdot \ln\left[\left(5 \pm 2\sqrt{6}\right)i\right]$

  $z = -i \cdot \left[\ln\left(5 \pm 2\sqrt{6}\right) + \ln i\right]$

  Note, from above, that $e^{i\pi} = -1$, so $e^{i\pi/2} = i$, and then: $\ln i = \frac{\pi}{2} i$

  $$z = -i \cdot \left[\ln\left(5 \pm 2\sqrt{6}\right) + \frac{\pi}{2} i\right] = \frac{\pi}{2} - i \cdot \ln\left(5 \pm 2\sqrt{6}\right)$$