Solutions to Math Club Challenge Questions
October 24, 2022

1) **What is the value of:** $1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}}$ **?**

Call the value given x. Add one. Divide 1 by the result. Add 1 again. The result is x. So,

$$\frac{1}{x+1} + 1 = x \quad \Rightarrow \quad \boldsymbol{x = \sqrt{2}}$$

2) **Pennies are placed on an 8 x 8 checkerboard in an alternating pattern of heads and tails. In each turn you flip over exactly two pennies that lie next to each other in the same row or column. Can you make a sequence of moves that leaves exactly one penny heads up? Explain.**

You start with 32 Heads (face up) and 32 Tails (face down).

Each move is one of the following:

- $HH \rightarrow TT$ a net change of -2 Heads
- $HT \rightarrow TH$ a net change of 0 Heads
- $TT \rightarrow HH$ a net change of $+2$ Heads

So, you start with an even number of Heads and every move a) increases the number of Heads by 2, b) decreases the number of Heads by 2, or c) leaves the number of Heads unchanged. Therefore, after every move you have an even number of Heads.

Result: You cannot create a sequence to leave only 1 Head at the end.

3) **(2 points) Dave calculated $16^8 = 4,294,967,294$. Without multiplying it out, tell whether Dave is right or wrong? Explain.**

Any two numbers ending in 6, when multiplied, end in another number ending in 6. So, 16^8 must end in a 6. Dave is wrong.

Note: $16^8 = 4{,}294{,}967{,}296$.

4) **Add the numbers 1 to 10,000.**

$$\begin{aligned} x &= 1 + 2 + \cdots + 9{,}999 + 10{,}000 \\ x &= 10{,}000 + 9{,}999 + \cdots + 2 + 1 \\ \hline 2x &= 10{,}001 \cdot 10{,}000 \end{aligned}$$

$$\boldsymbol{x = 50{,}005{,}000}$$

5) **Start with the general quadratic function: $y = ax^2 + bx + c$.**

   i. **What is the sum of the solutions to this equation?**

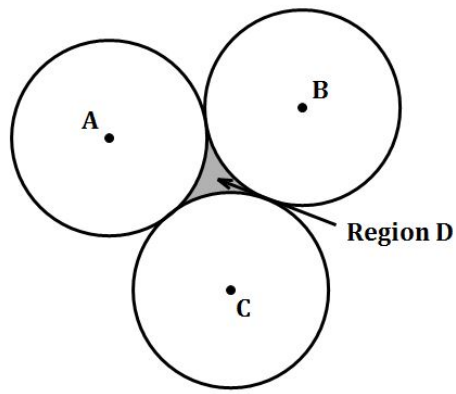
   ii. **What is the product of the solutions to this equation?**

The solutions to the quadratic are: $x = \left\{\frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a}\right\}$

i. $\dfrac{-b+\sqrt{b^2-4ac}}{2a} + \dfrac{-b-\sqrt{b^2-4ac}}{2a} = -\dfrac{\boldsymbol{b}}{\boldsymbol{a}}$

ii. $\dfrac{-b+\sqrt{b^2-4ac}}{2a} \cdot \dfrac{-b-\sqrt{b^2-4ac}}{2a} = \dfrac{b^2-(b^2-4ac)}{4a^2} = \dfrac{\boldsymbol{c}}{\boldsymbol{a}}$

6) **Three circles of radius 4 are tangent to each other as shown in the diagram. What is the area of Region D, bounded by all three circles? (Hint: draw segments connecting the centers of the circles.)**



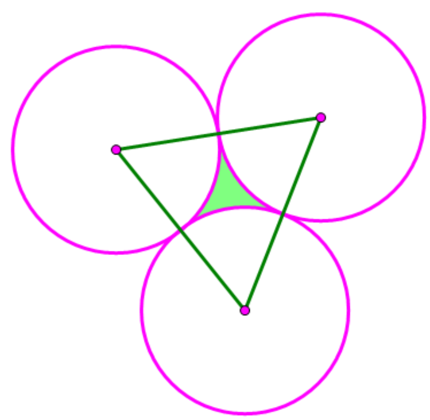
Solution: See the figure to the right.

The triangle shown is equilateral with base 8. It's height, then, is $4\sqrt{3}$. So, it's area is: $A = \frac{1}{2} \cdot 8 \cdot 4\sqrt{3} = 16\sqrt{3}$.

Each circle has a sector inside the triangle. Each sector has a central angle of 60°, for a total in all three circles of 180°.

So, the sum of the areas of the three sectors is $\frac{1}{2}$ the area of one of the circles. $A_{3\text{ sectors}} = \frac{1}{2} \cdot (\pi \cdot 4^2) = 8\pi$.

The area of the region bounded by the three circles is the difference of these two areas:



$\boldsymbol{A = 16\sqrt{3} - 8\pi}$