Partial Differentiation

Partial differentiation is differentiation with respect to a single variable, with all other variables being treated as constants.

Example 2.4: Consider the function $f(x, y) = xy + 2x + 3y$.

Full derivative:	**Partial derivative:**	**Partial derivative:**
$\frac{d}{dx}(xy + 2x + 3y) =$ $\left(x\frac{dy}{dx} + y\right) + 2 + 3\frac{dy}{dx}$	$\frac{\partial}{\partial x}(xy + 2x + 3y) =$ $y + 2$	$\frac{\partial}{\partial y}(xy + 2x + 3y) =$ $x + 3$

Notice in the partial derivative panels above, that the "off-variable" is treated as a constant.

- In the left-hand panel, the derivative is taken in its normal manner, including using the product rule on the xy-term.
- In the middle panel, which takes the partial derivative with respect to x, y is considered to be the coefficient of x in the xy-term. In the same panel, the $3y$ term is considered to be a constant, so its partial derivative with respect to x is 0.
- In the right-hand panel, which takes the partial derivative with respect to y, x is considered to be the coefficient of y in the xy-term. In the same panel, the $2x$ term is considered to be a constant, so its partial derivative with respect to y is 0.

Partial derivatives provide measures of rates of change in the direction of the variable. So, for a 3-dimensional curve, $\frac{\partial z}{\partial x}$ provides the rate of change in the x-direction and $\frac{\partial z}{\partial y}$ provides the rate of change in the y-direction. Partial derivatives are especially useful in physics and engineering.

Example 2.5: Let $w = x^2 e^{3y} \ln z + e^{4x} \sin(y + z) - \cos(xyz)$. Then,

$$\frac{\partial w}{\partial x} = 2xe^{3y} \ln z + 4e^{4x} \sin(y + z) + yz \sin(xyz)$$

$$\frac{\partial w}{\partial y} = 3x^2 e^{3y} \ln z + e^{4x} \cos(y + z) + xz \sin(xyz)$$

$$\frac{\partial w}{\partial z} = \frac{x^2 e^{3y}}{z} + e^{4x} \cos(y + z) + xy \sin(xyz)$$

Implicit Differentiation Using Partial Derivatives

Let $z = f(x, y)$. Then, the following formula is often a shortcut to calculating $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}$$

Let's re-do the examples from the previous pages using the partial derivative method.

Example 2.8: Find $\frac{dy}{dx}$ for the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 36$.

Let: $z = \frac{x^2}{4} + \frac{y^2}{9} - 36$. Then,

$$\frac{\partial z}{\partial x} = \frac{2x}{4} \qquad \frac{\partial z}{\partial y} = \frac{2y}{9} \qquad \frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = -\frac{\frac{2x}{4}}{\frac{2y}{9}} = -\frac{9x}{4y}$$

Example 2.9: Find $\frac{dy}{dx}$ for the equation: $x \sin y + y \cos x = 0$.

Let: $z = x \sin y + y \cos x$. Then,

$$\frac{\partial z}{\partial x} = \sin y - y \sin x \qquad \frac{\partial z}{\partial y} = x \cos y + \cos x$$

$$\frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = -\frac{\sin y - y \sin x}{x \cos y + \cos x} = \frac{y \sin x - \sin y}{x \cos y + \cos x}$$

Contrast the work required here with the lengthy efforts required to calculate these results on the two prior pages.

So, implicit differentiation using partial derivatives can be fast and, because fewer steps are involved, improve accuracy. Just be careful how you handle each variable. This method is different and takes some getting used to.

Comparison of Implicit Differentiation with Alternative (Calc 3) Formula

(problems from Section 3.8 of textbook)

13) $\sin xy = x + y$

Take derivatives:

$$\cos xy \cdot \frac{d}{dx}(xy) = 1 + \frac{dy}{dx}$$

$$\cos xy \cdot \left(x\frac{dy}{dx} + y\right) = 1 + \frac{dy}{dx}$$

$$x\cos xy\frac{dy}{dx} + y\cos xy = 1 + \frac{dy}{dx}$$

$$x\cos xy\frac{dy}{dx} - \frac{dy}{dx} = 1 - y\cos xy$$

$$(x\cos xy - 1)\frac{dy}{dx} = (1 - y\cos xy)$$

$$\frac{\partial z}{\partial y} \qquad\qquad -\frac{\partial z}{\partial x}$$

$$\boldsymbol{\frac{dy}{dx} = \frac{1 - y\cos xy}{x\cos xy - 1}}$$

Alternative Method

13) $\sin xy = x + y$

$$z = \sin xy - x - y$$

$$\frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = -\frac{y\cos xy - 1}{x\cos xy - 1} = \boldsymbol{\frac{1 - y\cos xy}{x\cos xy - 1}}$$

21) $6x^3 + 7y^3 = 13xy$

Take derivatives:

$$18x^2 + 21y^2\frac{dy}{dx} = 13\left(x\frac{dy}{dx} + y\right)$$

$$18x^2 + 21y^2\frac{dy}{dx} = 13x\frac{dy}{dx} + 13y$$

$$21y^2\frac{dy}{dx} - 13x\frac{dy}{dx} = 13y - 18x^2$$

$$(21y^2 - 13x)\frac{dy}{dx} = (13y - 18x^2)$$

$$\frac{\partial z}{\partial y} \qquad\qquad -\frac{\partial z}{\partial x}$$

$$\boldsymbol{\frac{dy}{dx} = \frac{13y - 18x^2}{21y^2 - 13x}}$$

21) $6x^3 + 7y^3 = 13xy$

$$z = 6x^3 + 7y^3 - 13xy$$

$$\frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = -\frac{18x^2 - 13y}{21y^2 - 13x} = \boldsymbol{\frac{13y - 18x^2}{21y^2 - 13x}}$$