Value of a Definite Integral as the Limit of a Riemann Sum

Process:

Start with the definition of a Riemann Sum with n sub-intervals of constant width, Δx:

$$S = \sum_{k=1}^{n} f(x_k^*) \cdot \Delta x$$

The total width of the interval in the integral is $b - a$. If we need to fit n sub-intervals of constant width in $b - a$, the width of each interval is: $\Delta x = \frac{integral\ interval\ width}{number\ of\ sub-intervals} = \frac{b-a}{n}$.

The x_k^* values are based on the type of Riemann Sum we use. Typically, a *right-endpoint Riemann Sum* is used in these calculation. Therefore, the x_k^* values are the x-values at the right endpoints of the subintervals (see the illustration below), i.e.:

$$x_1^* = a + 1(\Delta x), \quad x_2^* = a + 2(\Delta x), \quad x_3^* = a + 3(\Delta x), \quad \dots, \quad x_n^* = a + n(\Delta x) = b$$

When a set of values is defined in terms of an indexed sequence (e.g., $1, 2, 3, \dots$), it is common to refer to the sequence in terms of an index variable, such as i or k. The above x-values can be described in index form as $x_k^* = a + k\ (\Delta x)$, where k takes on the values: $1, 2, 3, \dots, n$. Note that the final value of x is the upper limit of integration, i.e., $x_n^* = b$.

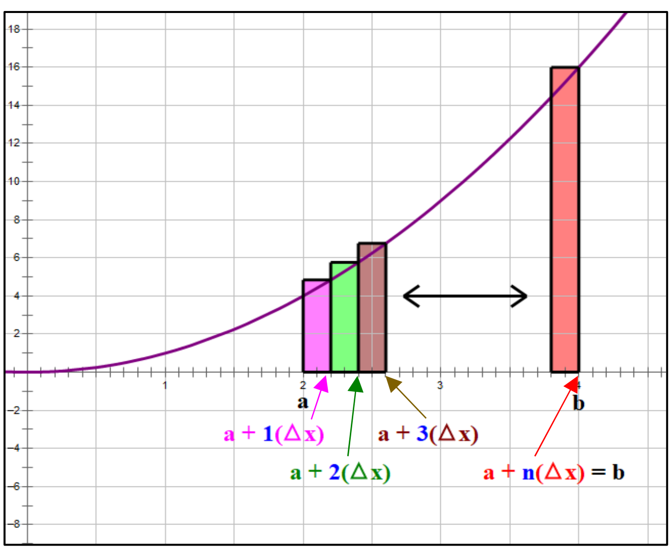


Illustration of Right-Endpoint Riemann Sum Formula

To obtain the area of the rectangles in the Riemann sum, we multiply the width of each subinterval, Δx, by its height, which is the function value at the right endpoint of the interval, $f[a + k(\Delta x)]$, and add them all up.

In formula terms, that is:

$$\sum_{k=1}^{n} \underbrace{f[a + k(\Delta x)]}_{\text{rectangle height}} \cdot \Delta x$$

rectangle width

Finally, to obtain the value of the integral, we increase the number of sub-intervals, n, to approach ∞:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f[a + k(\Delta x)] \cdot \Delta x$$

In practice, we substitute expressions for $f[a + k(\Delta x)]$ and for Δx in the final solution to the problem. For an illustration of the entire process, see the example on the next page.

Example: Provide the exact value of $\int_2^4 x^2\,dx$ by expressing it as the limit of a Riemann sum.

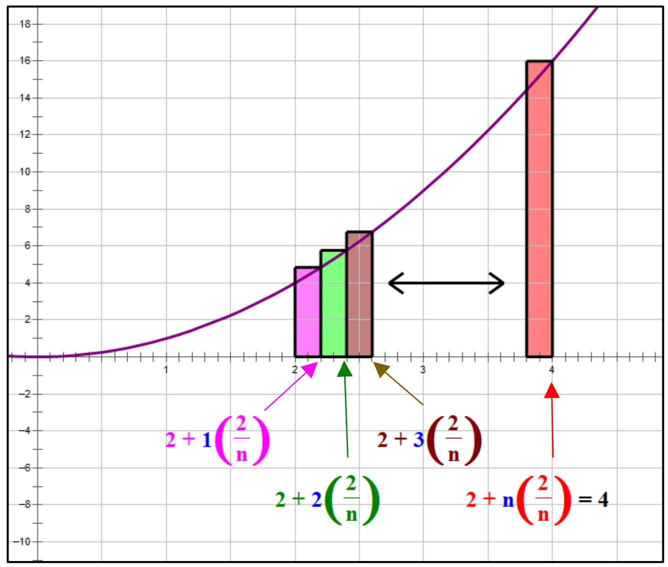
Start with the definition of a Riemann Sum with n sub-intervals of constant width:

$$S = \sum_{k=1}^{n} f(x_k^*) \cdot \Delta x$$

Using a *right-endpoint Riemann Sum* with n subintervals, we want to find the constant width of each sub-interval, Δx. Note that the limits of integration are from $a = 2$ to $b = 4$.

$$\Delta x = \frac{b-a}{n} = \frac{4-2}{n} = \frac{2}{n}$$

So, each subinterval is $\frac{2}{n}$ wide.



Next, we want to find the heights of the rectangles in the Riemann Sum, which are the function values at the right endpoints of the sub-intervals.

The x-values of the *right endpoints* of the subintervals, in index notation, are:

$$x_k^* = a + k(\Delta x), \quad k = 1, 2, \dots, n$$

Substituting in the values of a and Δx, we get:

$$x_k^* = 2 + k\left(\frac{2}{n}\right) = \left(2 + \frac{2k}{n}\right), \quad k = 1, 2, \dots, n$$

In this problem, $f(x) = x^2$. So, at the right endpoints of the subintervals, the rectangle heights are:

$$f(x_k^*) = f\left(2 + \frac{2k}{n}\right) = \left(2 + \frac{2k}{n}\right)^2$$

Finally, the integral as a limit of the *right-endpoint Riemann Sum* is given by:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*) \cdot \Delta x$$

which becomes:

$$\int_2^4 x^2 dx = \lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{\left(2 + \frac{2k}{n}\right)^2}_{\text{rectangle height}} \underbrace{\left(\frac{2}{n}\right)}_{\text{rectangle width}} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(4 + \frac{8k}{n} + \frac{4k^2}{n^2}\right) \cdot \left(\frac{2}{n}\right)$$

Note that we could have used the limit of a left-endpoint Riemann Sum, a midpoint Riemann Sum, or even a trapezoidal Riemann Sum to obtain the value of the integral. The formulas for these alternatives would look a little different, but the limit and, therefore, the value of the integral, would be the same.