GenSynD.wpd

GENERALIZING SYNTHETIC DIVISION (GenSynD)

We present a method of doing division of polynomials that is **much faster, easier and less error prone** than the "classical" method.

It just involves **three** steps repeated as many times as necessary to complete the division. The format puts the quotient and remainder in the positions where we expect them from dealing with the other division algorithms that we have seen. The reason that it is easier and faster is that we use the **short** form for the polynomials and instead of doing a repeated subtraction algorithm, we do a "reversing cross multiplication" algorithm.

The steps are **Estimate, Multiply** and **Add** (**EMA**). These steps are repeated for each column required in the quotient of the division. (For those who are interested in their grade point average they can use Guess, Product and Add (**GPA**) instead.

We mention that even the "classical" approach can be made much easier and faster by using the **short form** for polynomials instead of the expanded form. This is due to the fact that **polynomials** are basically **arithmetic in an unknown base**. Hence we are entitled to write them in short form.

For example if we are doing

$$x^4-3x^3+5x^2-4x+7 \text{ divided by } x^2-2x+5$$

this becomes $\mathbf{1\ \bar{3}\ 5\ \bar{4}\ 7}$ divided by $\mathbf{1\ \bar{2}\ 5}$ using **place value.**

This makes the division much easier to write and work with. Be sure to remember that in all the operations on polynomials we **cannot** carry or borrow between columns since we don't know how many it takes to cause a carry or how many are obtained from a borrow. In base 10 it takes 10 to cause a carry. In an unknown **base it would take x to cause a carry**. But x is **unknown**. Hence we **can't carry**.

Note that when dividing by a polynomial with a **leading coefficient** greater than **one** in the divisor, if the problem isn't "**rigged**" then fractions (and/or decimals) can occur.

These cause the arithmetic to get **really messy by hand.** Hence in nearly all textbooks and problem sets the problems are rigged so that the coefficients in the quotient (and usually the remainder also) are integers. We will do likewise in our examples to follow. The algorithm still works when it involves fractions/decimals, but it becomes quite a mess if doing the calculations by hand.

Consider $(10x^3-x^2-18x+1)/(5x-3)$.

<u>THE SET UP</u>:	**opposite** →		**3**			**LEGEND:**
						Multiply these two lines.
Estimator in parens:	**(5) $\bar{3}$**	**10**	**$\bar{1}$**	**$\overline{18}$**	**1**	**Add these two lines.**
Bring down the 10 to start:		**10**				**Estimate up to vertical bar.**

The finished division looks like this:

		3			
	2	**1**	**$\bar{3}$**		**M**
(5) $\bar{3}$	**10**	**$\bar{1}$**	**$\overline{18}$**	**1**	
		6	**3**	**$\bar{9}$**	**A**
	10	**5**	**$\overline{15}$**	**$\bar{8}$**	**E**

The quotient is **2 1 $\bar{3}$ = $2x^2+x-3$** and is where expected.
The remainder is the constant **$\bar{8}$** and is where expected.
We write the **opposite** of **$\bar{3}$** <u>at the top</u> so that we get to **<u>add</u>** the two lines indicated **<u>rather than subtract</u>**. It's easier.

Notice that the only writing done besides the setup and the answer is the five numbers six, three, ninebar, five and fifteenbar. The amount of **extra writing** is **quite minimal** compared to the "normal" way of doing division. Also some folks don't write these five numbers but do them in their head thus writing just the quotient and remainder. That should catch the eye of those who only know the "normal" division algorithm! After getting used to the algorithm one can leave off the legend.

We will show the cycles for completing the problem above.

Each cycle is: **Estimate, Multiply and Add.**

Est: **Divide** the estimator **5** into the **10** brought down. **10/5=2** so write the **2** above the top horizontal line.

Multiply: Multiply the **2** we just wrote times the **3** at the top and write this result <u>beneath the **onebar**</u>.

Add: **Add** the **6** we just wrote to the **onebar** above it and record this result **5** on the <u>bottom line</u>.

This **completes** one cycle of **EMA**. Each cycle begins with an estimation then does a multiplication and then an addition with the numbers recorded in a like manner as the first cycle.

We obtain **5/5=1** then **3*1=3** and then **$\bar{1}\bar{8}$+3=$\bar{1}\bar{5}$** for the <u>second</u> cycle.

We obtain **$\bar{1}\bar{5}$/5=$\bar{3}$** then **3*$\bar{3}$=$\bar{9}$** and then **1+$\bar{9}$=$\bar{8}$** for the <u>last</u> cycle.

The **complete problem** is just **<u>nine</u>** <u>signed number</u> operations and the writing is quite minimal. When one gets really used to the algorithm, then it takes about 15 seconds to work the problem once it is set up.

Here is another problem. **$(12x^3-21x^2+30x+9)/(12x+3)$**

		$\bar{3}$		
	1	$\bar{2}$	3	
(12) 3	**12**	**$\bar{2}\bar{1}$**	**30**	**9**
		$\bar{3}$	**6**	**$\bar{9}$**
	12	**$\bar{2}\bar{4}$**	**36**	**0**

cycle	**1**	**2**	**3**
E:	**12/12=1**	**$\bar{2}\bar{4}$/12=$\bar{2}$**	**36/12=3**
M:	**$\bar{3}$*1=$\bar{3}$**	**$\bar{3}$*$\bar{2}$=6**	**$\bar{3}$*3=$\bar{9}$**
A:	**$\bar{3}$+$\bar{2}\bar{1}$=$\bar{2}\bar{4}$**	**30+6=36**	**9+$\bar{9}$=0**

The quotient is then **x^2-2x+3** with remainder **zero**.

Next we consider dividing by polynomials **ax^2+bx+c (a<>0)**. (Recall that a<>0 means a is not equal to zero.) This will require **2x2** cross multiplication for all the **multiplications except** the first and last. Recall that **2x2 cross** multiplication (the **middle** column) multiplies the <u>first</u> number of the <u>first</u> row times the **second** number in the **second** row and then adds this to the product of the <u>second</u> number in the <u>first</u> row by the **first** number in the **second** row. For **example**

2 $\bar{3}$
4 5 yields 2*5+4*$\bar{3}$ = 10+$\bar{1}\bar{2}$ = $\bar{2}$

Multiplication <u>is the **only thing** that changes</u> for these larger divisors. The estimations and additions work essentially the same for any of these divisions.

Now let's do this division: **$(6x^5+\bar{5}x^4+0x^3+2x^2+\bar{8}x+2)/(3x^2+\bar{1}x+4)$**

<u>Notice the **zero x cubed** term.</u> It is not necessary to write in

the expanded form but it is **critical** to write the **zero** in the **short form**, otherwise the exponents to the left of this zero times **x cubed** term **get messed up** (they will be one smaller than they should be.)

			1	$\bar{4}$				
		2	$\bar{1}$	$\bar{3}$	1			**M**
(3)	$\bar{1}$ 4	6	$\bar{5}$	0	2	$\bar{8}$	2	**A**
			2	$\bar{9}$	1	13	$\bar{4}$	
		6	$\bar{3}$	$\bar{9}$	3	5	$\bar{2}$	**E**

The quotient is $2x^3-x^2-3x+1$ and the remainder is $5x-2$.

Cycle	**1**	**2**	**3**	**4**	**for the remainder**
Estimate:	**6/3=2**	$\bar{3}/3=\bar{1}$	$\bar{9}/3=\bar{3}$	**3/3=1**	**(no Est)**
Multiply:	**1*2=2**	$\bar{1}*1+2*\bar{4}=\bar{9}$	$1*\bar{3}+\bar{1}*\bar{4}=1$	$1*1+\bar{3}*\bar{4}=13$	$\bar{4}*1=\bar{4}$
Add:	$\bar{5}+2=\bar{3}$	$0+\bar{9}=\bar{9}$	**2+1=3**	$\bar{8}+13=5$	$2+\bar{4}=\bar{2}$

Notice that the **first** and **last** products are **single** and the rest of them are **2x2** cross products. There are **19** signed number operations done to complete this division. If you are 90% accurate at these operations, you will most likely make **TWO** mistakes while working this problem. It is **absolutely essential** that one reaches **99+%** accuracy in these little operations to stand a chance of succeeding in these divisions (and in algebra in general).

It is **much easier** to catch on to this division routine if you **see** it **done in person**. But hopefully this written account will get you up to speed.

			$\bar{1}$	$\bar{1}$			
		2	0	0	1		
(9)	1 1	18	2	2	9	2	2
			$\bar{2}$	$\bar{2}$	0	$\bar{1}$	$\bar{1}$
		18	0	0	9	|	|

This is $(18x^5+2x^4+2x^3+9x^2+2x+2)/(9x^2+x+1)$ for which the quotient is $2x^3+1$ and the remainder is the twin towers $(x+1)$.

It is quite easy to make up **new** problems that do not get into

fractions. All one has to do is to start with a number in the dividend that divides by the estimator and then for each succeeding column of the dividend choose a number that causes the sum to be divisible by the estimator.

For example in the problem above we chose the first column (the leftmost column) to be **18** which is divisible by **9**. Then we chose **2** for the next column so that the sum was **0**. We could have just as well chosen **-7, 11, 20** or any other number whose remainder when dividing by **9** is **2**. Then the sum would divide by **9** with no remainder. I've made many pages of problems for my students using this technique.

GenSynD can be used with **higher dimension** divisors. This only requires that the products involve higher cross products. For example, dividing by a **cubic** polynomial will require products of the form **1x1, 2x2, 3x3, 3x3,...,3x3, 2x2, 1x1** the last three products of which are used in the calculation of the remainder.

A **slight adaptation** of this division algorithm provides an **algorithm for taking square roots** of polynomials. To avoid fractions make sure the leftmost column of the polynomial is a perfect square. Or create a problem by squaring a polynomial.

Twice this first column's square root becomes the **estimator 6** for the process. And the **first (leftmost) column of the answer** is **that square root 3** (**not** twice the square root). **Bring down** the second column **6bar** and divide it by the estimator **6**. This result **onebar** is the **next column** of the answer. **Above it** put its **opposite 1**.

Then **multiply** $\bar{1}$***1** and add this $\bar{1}$ to **13** to get **12**. Divide **12** by the **estimator** and put the **2** at the **top** in the third column. Put its **opposite 2bar** above it and do the **2x2** cross product to get **4**. Add **4** to the next column to get **0**. Then multiply the **2** by **2bar** to get **4bar** and add this to the **4** to get the second **0**. The original square root **3** of the first column is **never used again**. It just provides us with the estimator.

The process looks like this for $(3x^2-x+2)^2 = 9\ \bar{6}\ 13\ \bar{4}\ 4$:

Estimator = 2*sqr(9)=6		1	$\bar{2}$		
sqr(9) ⟶	3	$\bar{1}$	2		
Estimator (6) √	9	$\bar{6}$	13	$\bar{4}$	4
			$\bar{1}$	4	$\bar{4}$
		$\bar{6}$	12	0	0

The products are $1*\bar{1}=\bar{1}$ and then $1*2+\bar{1}*\bar{2}=4$ then $2*\bar{2}=\bar{4}$. The numbers for the products are **generated AS** the process goes on. The last two columns need not produce a zero remainder. Instead of fourbar and four we could have any numbers we wish. Then the polynomial would not necessarily be a perfect square.

For a longer problem whose square root is a four column polynomial we would have **1x1, 2x2, 3x3, 2x2, 1x1** cross products. Squaring a polynomial always produces an **odd** number of columns so we draw the **vertical line** just to the **right of the center column** for the remainder calculation.

If we continue estimating **past the vertical line** in the division or in the square root algorithm, then we would have a **decimal point** in place of the vertical bar at the top. Then we are creating **"decimal" polynomials.** Continuing forever would create an **infinite series.**

As an example of a **"decimal" polynomial** consider

$$2\ \bar{1}\bar{3}\ 4\ .\ \bar{7}\ 15\ \bar{6} = 2x^2 + \bar{1}\bar{3}x + 4 + \bar{7}/x + 15/x^2 + \bar{6}/x^3$$

Try $1/(x^2-x-1)$ to generate the Fibonacci sequence **1 1 2 3 5...**

1/89 does the same thing in arithmetic **IF** we do no carrying. Any polynomial division where the estimator is **1** generates a similar type phenomenon. The higher the degree of the divisor, the larger the linear combination producing the next number in the sequence.

And **don't forget** that we can **check** addition, subtraction, multiplication, division, squaring and square roots in polynomial algebra by **CASTING OUT NINES** --- subject of another article.

Let's look at one more example of taking square roots.

Consider: $\mathbf{16x^6-8x^5-15x^4+12x^3+2x^2-4x+1}$

$$= 16x^6+\bar{8}x^5+\overline{15}x^4+12x^3+2x^2+\bar{4}x+1$$

$$\begin{array}{r|cccc|ccc}
 & & 1 & 2 & \bar{1} & & & \\
\text{sqr(16)} \longrightarrow & 4 & \bar{1} & \bar{2} & 1 & & & \\
\hline
\text{Estimator:(8)} & 16 & \bar{8} & \overline{15} & 12 & 2 & \bar{4} & 1 \\
 & & & \bar{1} & \bar{4} & \bar{2} & 4 & \bar{1} \\
\hline
 & & \bar{8} & \overline{16} & 8 & 0 & 0 & 0
\end{array}$$

So the square root is $\mathbf{4x^3-x^2-2x+1}$ with remainder **zero.**
The estimator **8** is **twice** the **<u>square root</u>** of the leading **16.**

Bring down the **eightbar** and divide it by the estimator to get the **1bar** to the right of the **4.**

Put **1** above the **1bar** and multiply the **1** and **1bar** to obtain the **1bar** beneath the **15bar.**

Add the **1bar** and the **15bar** to get the **16bar.**

The estimator goes **2bar** times so write the **2bar** above the **15bar.**

Write a **2** above this **2bar.** Cross multiply the $\bar{1}\ \bar{2}$ and **1 2** to obtain the **4bar** beneath the **12.**

Add the **12** and **4bar** to get the **8** beneath the **4bar.**

Estimate **8** into this **8** to get the **1** above the **12.**

Put a **1bar** above this **1.** Then do **3x3, 2x2 and 1x1** cross multiplications to get the **2bar, 4 and 1bar** to be added to the **2, 4bar and 1** to obtain the remainder **0 0 0.**

This **zero remainder** means that the original polynomial is a **perfect square.** If we had some other numbers instead of the last **2, 4bar and 1,** then the remainder would not be **zero.** When the polynomial is a **perfect square** we can also do the algorithm with the sequence of digits **reversed.**

Thus we could do the algorithm on $\mathbf{1\ \bar{4}\ 2\ 12\ \overline{15}\ \bar{8}\ 16}$ to obtain the **reversed** square root $\mathbf{1\ \bar{2}\ \bar{1}\ 4}$.

If the estimator **does not go nicely** into the number on the bottom line (in division or in square roots) then **fractions and/or decimals** occur. The reason for this is that in base x we **cannot carry** the remainder of the estimation into the next column. This makes the process **quite nasty** when doing the work **by hand.** Hence nearly all the problems in texts are nicely **rigged** to make these estimations come out integral.

A HANDY *HINT* FOR LEARNING THESE ALGORITHMS:

Once you have worked through an example. Go through the **same** example **several** more times -- enough times so that you have practically memorized the example. Then progress to another example and do **likewise.** This impresses upon the mind the **flow** of the algorithm with minimal effort.

A similar technique is used in learning a language. Start at the **end** of the sentence or phrase and practice the **last few words.** Then add in a few more words and practice that to the end. When the first few words of the sentence or phrase are reached it is easy to do the whole thing. You might try this the next time you are trying to learn a language or when you are trying to memorize a passage.

When learning to drive a **manual transmission** it is best to stick with the **same** auto until you get smooth at it. It would be hard to get it down if each session was in a different auto with a different shift pattern (on the floor, the steering column, etc.)

The next page gives some division problems with answers and cast out digits when checking by casting out nines.

Div2$\bar{3}$.wpd

NAME:______________________ DATE:________

1) $(4\ \bar{8}\ \bar{1}\ 9)/(2\ \bar{3})$

2) $(3\ \bar{5}\ \bar{2}\ \bar{4})/(3\ 1)$

3) $(6\ 1\ \bar{9}\ \bar{7})/(3\ \bar{4})$

4) $(11\ \bar{4}\ \bar{7}\ \overline{11}\ \bar{5})/(11\ 7)$

5) $(2\ 7\ \overline{12}\ 8)/(1\ 5)$

6) $(3\ \bar{5}\ \bar{4}\ 4)/(\bar{3}\ 2)$

7) $(2\ 6\ \bar{9}\ 3\ 40)/(\bar{1}\ \bar{4})$

8) $(13\ \overline{12}\ \bar{1}\ 3)/(13\ 1)$

9) $(6\ \bar{7}\ \bar{7}\ 1)/(2\ \bar{3})$

10) $(2\ 13\ \bar{6}\ 7)/(1\ 7)$

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11) $(3\ \bar{8}\ \bar{4}\ 8\ \bar{5})/(3\ \bar{2}\ 1)$

12) $(5\ \bar{7}\ 9\ 3\ \bar{6})/(5\ \bar{2}\ \bar{3})$

13) $(8\ \bar{4}\ \bar{2}\ \bar{3}\ \bar{6})/(4\ 0\ 3)$

14) $(14\ \bar{5}\ 0\ 3\ \bar{4})/(7\ 1\ \bar{3})$

15) $(12\ \bar{1}\ \overline{14}\ 7\ \bar{3})/(12\ \bar{1}\ \bar{2})$

16) $(6\ 4\ \bar{8}\ 5\ \bar{1})/(6\ \bar{2}\ 0)$

17) $(2\ \bar{7}\ 0\ \bar{1}\ 3)/(1\ \bar{4}\ 3)$

18) $(6\ \bar{3}\ 9\ 4\ \bar{7})/(3\ 0\ 0)$

19) $(4\ \bar{4}\ \overline{11}\ 5\ \bar{4})/(2\ 1\ \bar{3})$

20) $(2\ \bar{9}\ \overline{14}\ 11\ 14\ 3)/(1\ \bar{5}\ \bar{3})$

Answers: Q:Quotient R:Remainder COD=Cast Out Nines Digit

1) Q: $2\ \bar{1}\ \bar{2}$ R: 3 COD=4

2) Q: $1\ \bar{2}\ 0$ R: $\bar{4}$ COD=1

3) Q: $2\ 3\ 1$ R: $\bar{3}$ COD=0

4) Q: $1\ \bar{1}\ 0\ \bar{1}$ R: 2 COD=2

5) Q: $2\ \bar{3}\ 3$ R: $\bar{7}$ COD=5

6) Q: $\bar{1}\ 1\ 2$ R: 0 COD=7

7) Q: $\bar{2}\ 2\ 1\ \bar{7}$ R: 12 COD=6

8) Q: $1\ \bar{1}\ 0$ R: 3 COD=3

9) Q: $3\ 1\ \bar{2}$ R: $\bar{5}$ COD=2

10) Q: $2\ \bar{1}\ 1$ R: 0 COD=7

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11) Q: $1\ \bar{2}\ \bar{3}$ R: $4\ \bar{2}$ COD=3

12) Q: $1\ \bar{1}\ 2$ R: $4\ 0$ COD=4

13) Q: $2\ \bar{1}\ \bar{2}$ R: $0\ 0$ COD=2

14) Q: $2\ \bar{1}\ 1$ R: $\bar{1}\ \bar{1}$ COD=8

15) Q: $1\ 0\ \bar{1}$ R: $6\ \bar{5}$ COD=1

16) Q: $1\ 1\ \bar{1}$ R: $3\ \bar{1}$ COD=6

17) Q: $2\ 1\ \bar{2}$ R: $\overline{12}\ 9$ COD=6

18) Q: $2\ \bar{1}\ 3$ R: $4\ \bar{7}$ COD=0

19) Q: $2\ \bar{3}\ \bar{1}$ R: $3\ \bar{7}$ COD=8

20) Q: $2\ 1\ \bar{3}\ \bar{1}$ R: $0\ 0$ COD=7