# Generalized Remainder Theorem (GRT)

Suppose $D(x)=(x-r_1)(x-r_2)...(x-r_n)$ divided into **P(x)** gives quotient **Q(x)** and remainder **R(x)**. (degree **D(x)** ≤ degree **P(x)**).

Then for **i=1 to n**, $P(r_i)= R(r_i)$.

The theorem is easy to prove. Let the divisor be called **D(x).**

Then $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$ and multiplying through by **D(x)**

gives $P(x) = Q(x)D(x) + R(x)$.

Therefore for the values of **x** that make **D(x)=0** we have **P(x)=Q(x)*0 + R(x)**; that is, **P(x)= R(x)**.
This works for both **real** and **complex** values of **x**.

The division of **P(x)** by **D(x)** can be done easily using the generalization of Synthetic Division (**GenSynD** algorithm).

**Example 1)** Evaluate $P(x)= 2x^4-5x^3-4x^2+x+3$ **at x=3 and x=-1.**
$D(x) = (x-3)(x+1) = x^2-2x-3$ divided into **P(x)** gives **R(x)=-2x+3**
so **P(3)= R(3)= -2(3)+3=-3** and **P(-1)=R(-1)= -2(-1)+3=5**.

**Example 2)** Evaluate $P(x)= x^4-2x^3+3x^2+3x-6$ **at x=i and x=-i.**
$D(x) = (x+i)(x-i) = x^2+1$ divided into **P(x)** gives **R(x)=5x-8** so **P(i)=R(i)= 5i-8** and **P(-i)=R(-i)= -5i-8**.

**Example 3)** Evaluate $P(x)= 2x^3-2x^2+5x+70$ **at x=2+3i**.
Let $D(x) = x^2-4x+13$ which has roots **2+3i** and **2-3i**. Then **P(x)** divided by **D(x)** has remainder **R(x)= 3x-8**. So

**P(2+3i) = R(2+3i) = 3(2+3i)-8 = -2+6i**.

We also have **P(2-3i) = R(2-3i) = 3(2-3i)-8 = -2-6i**.

In general to evaluate **P(x)** at **the complex value a+bi** we form

$$D(x)=(x-(a+bi))(x-(a-bi)) = x^2-2ax+(a^2+b^2).$$

Then dividing **P(x)** by **D(x)** we get the corresponding remainder **R(x).** Since this is a linear expression, **P(a+bi) = R(a+bi)** and **P(a-bi) = R(a-bi)** are **complex conjugates**. Thus after we evaluate **R(a+bi)** we just change the sign of the imaginary part of its answer to get the value for **R(a-bi).**

**Example 4)** Evaluate **P(x)= $3x^5-5x^4+2x^3+3x^2+7x-9$ at x=1-i.**

**Form D(x)= (x-(1-i))(x-(1+i))= $x^2-2x+2$.**

Then P(x) divided by D(x) has remainder **R(x)= 5x-3.** So

**P(1-i)=R(1-i)=5(1-i)-3 = 2-5i**

Also **P(1+i)=R(1+i)=5(1+i)-3 = 2+5i.**

The division **P(x)/D(x)** using **GenSynD** is:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | **2** | **$\bar{2}$** | | | |
| | | | **3** | **1** | **$\bar{2}$** | **$\bar{3}$** | | |
| **(1)** | **$\bar{2}$** | **2** | **3** | **$\bar{5}$** | **2** | **3** | **7** | **$\bar{9}$** |
| | | | | **6** | **$\bar{4}$** | **$\bar{6}$** | **$\bar{2}$** | **6** |
| | | | **3** | **1** | **$\bar{2}$** | **$\bar{3}$** | **5** | **$\bar{3}$** |

**GRT** gives us a way to evaluate polynomials at **complex** values **WITHOUT** having to work with complex values in the division of **P(x)** by **D(x).** It keeps all the calculations **real** until the **very end** when evaluating **R(x)** with **x** complex. This way we also avoid having to substitute the complex number into the higher powers of **x** if we try to evaluate **P(x)** directly.

Of course there is not much advantage to evaluating with real roots this way since we can just divide P(x) by x-a and look at the remainder. But we could do just **ONE division** to obtain the evaluations for **two (or more)** real roots using this generalization.

Also if **R(x)=0** then **D(x),** the **factors** of **D(x),** and **Q(x)** are all **factors of P(x)**, which is the factor theorem with higher degree divisor.