Trigonometry – Hard Problems

Solve the problem.

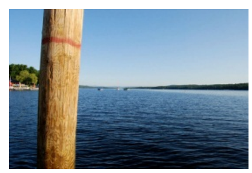
1) A surveyor is measuring the distance across a small lake. He has set up his transit on one side of the lake 90 feet from a piling that is directly across from a pier on the other side of the lake. From his transit, the angle between the piling and the pier is 35°. What is the distance between the piling and the pier to the nearest foot?

This problem is very difficult to understand. Let's see if we can make sense of it. Note that there are multiple interpretations of the problem and that they are all unsatisfactory.

- The problem does not say the lake is a circle, but if it is not, the problem cannot be solved. So, let's assume the lake is a circle.
- What is a transit? From www.surveyhistory.org, we get "The transit is used by the surveyor to measure both horizontal and vertical angles."

A transit →

A piling →



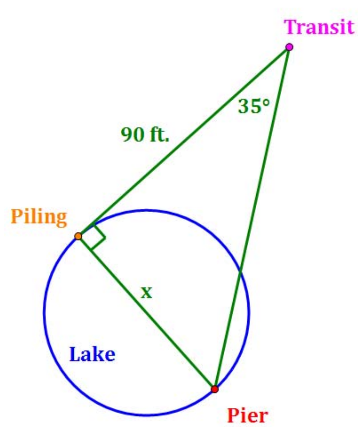
- What is a piling? From wisegeek.com, a piling is "a component in a foundation which is driven into the ground to ensure that the foundation is deep."
- We will need a right angle to solve a problem with only one length and angle given, so let's infer a 90˚ angle as shown in the following diagram.

Based on the diagram, which meets every criterion laid out in the problem, we now have:

$$\tan 35° = \frac{x}{90}$$

$$x = 90 \cdot \tan 35° = \mathbf{63\ ft.}$$

The body of water in this problem would be better described as a pond than a lake!

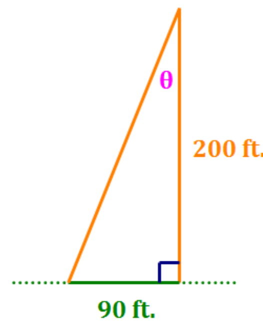


2) A building 200 feet tall casts a 90 foot long shadow. If a person looks down from the top of the building, what is the measure of the angle between the end of the shadow and the vertical side of the building (to the nearest degree)? (Assume the person's eyes are level with the top of the building.)

Based on the illustration at right, we get the following:

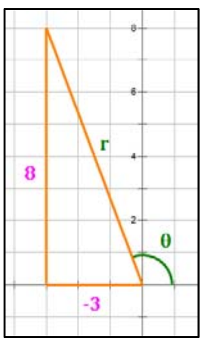
$$\tan\theta = \frac{90}{200} = .45$$

$$\theta = \tan^{-1}(.45) = \mathbf{24°}$$



Find the exact value of the indicated trigonometric function of θ.

3) $\tan\theta = -\frac{8}{3}$, θ in quadrant II Find cos θ.



The key on this type of problem is to draw the correct triangle.

Recall that $\tan\theta = \frac{y}{x} = -\frac{8}{3}$

Since θ is in $Q2$, we can plot the point $(-3, 8)$ to help us.

Notice that the hypotenuse must have length: $r = \sqrt{(-3)^2 + (8)^2} = \sqrt{73}$.

Then, $\cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{73}} = \frac{-3\sqrt{73}}{73}$

Find the exact value of the expression, if possible. Do not use a calculator.

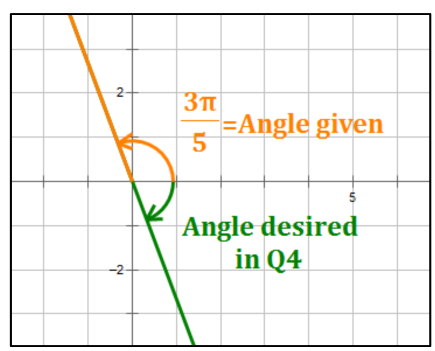
4) $\tan^{-1}\left[\tan\left(\frac{3\pi}{5}\right)\right]$

The angle $\frac{3\pi}{5}$ is in Q2, but tangent is defined only in Q1 and Q4. Further, $\tan\frac{3\pi}{5} < 0$ in Q2.

So we seek the angle in Q4, where tangent is also < 0, with the same tangent value as $\frac{3\pi}{5}$.

Recall that the tangent function has a period of π radians. Then,

$$\tan^{-1}\left(\tan\frac{3\pi}{5}\right) = \frac{3\pi}{5} - \pi = -\frac{2\pi}{5}$$



5) Find the exact value of the expression.

a. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

b. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

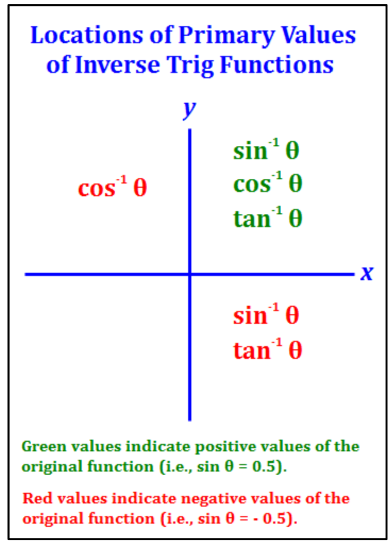
c. $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$

Know where the primary values for the inverse trig functions are defined.

$\sin^{-1}\theta$ is defined in Q1 and Q4.

$\cos^{-1}\theta$ is defined in Q1 and Q2.

$\tan^{-1}\theta$ is defined in Q1 and Q4.



Using a calculator, solve the following problems. Round your answers to the nearest tenth.

6) A ship leaves port with a bearing of N 63° W. After traveling 25 miles, the ship then turns 90° and travels on a bearing of S 27° W for 24 miles. At that time, what is the bearing of the ship from port?

You can think of this as adding two vectors with the bearings identified in the problem.

Step 1: Convert each vector into its **i** and **j** components, using reference angles.

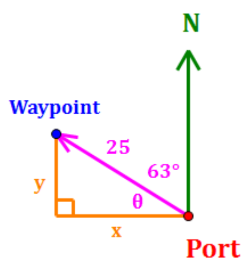
Let **u** be a vector of 25 mi. at bearing: N63°W

From the diagram at right,

$\theta = 90° - 63° = 27°$

$x = -25\cos 27° = -22.2752$

$y = 25\sin 27° = 11.3498$



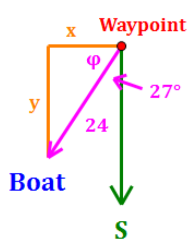
Let **v** be a vector of 24 mi. at bearing: S27°W

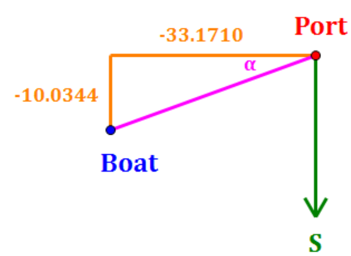
From the diagram at right,

$\varphi = 90° - 27° = 63°$

$x = -24\cos(63°) = -10.8958$

$y = -24\sin(63°) = -21.3842$





Step 2: Add the results for the two vectors

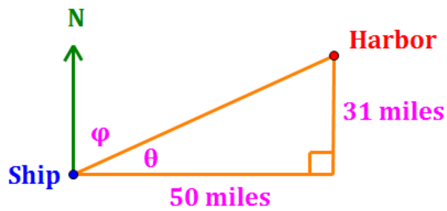
$$\begin{aligned} \mathbf{u} &= \langle -22.2752,\ 11.3498 \rangle \\ \mathbf{v} &= \langle -10.8958, -21.3842 \rangle \\ \hline \mathbf{u}+\mathbf{v} &= \langle -33.1710, -10.0344 \rangle \end{aligned}$$

Step 3: Find the resulting angle and convert it to its "bearing" form (see diagram above right).

$$\alpha = \tan^{-1}\left(\frac{-10.0344}{-33.1710}\right) = 16.8°$$

Bearing $= \text{S}\ (90° - 16.8°)\ \text{W} = \mathbf{S\ 73.2°\ W}$

7) A ship is 50 miles west and 31 miles south of a harbor. What bearing should the Captain set to sail directly to harbor?



$$\theta = \tan^{-1}\left(\frac{31}{50}\right) = 31.8°$$

$$\varphi = 90° - 31.8° = 58.2°$$

$$\textbf{Bearing} = \textbf{N } 58.2° \textbf{ E}$$

For Problems 8 and 9 , use: $f = \frac{\omega}{2\pi} = \frac{1}{\text{period}}$ or $\boldsymbol{\omega = 2\pi f}$ with $\omega > 0$

Harmonic motion equations: $d = a\cos\omega t$ or $d = a\sin\omega t$

An object is attached to a coiled spring. The object is pulled down (negative direction from the rest position) and then released. Write an equation for the distance of the object from its rest position after t seconds.

8) amplitude = 8 cm; period = 5 seconds

The spring will start at a y-value of -8, and oscillate from -8 to 8 over time. A good representation of this would be a cosine curve with lead coefficient $= -8$, because $\cos(0) = 1$, whereas, $\sin(0) = 0$.

The period of our function is 5 seconds. So, we get:

$$f = \frac{1}{\text{period}} = \frac{1}{5} \quad \textit{and} \quad \omega = 2\pi f = 2\pi \cdot \frac{1}{5} = \frac{2\pi}{5}$$

The resulting equation, then, is: $\mathbf{d = -8 \cos\left(\frac{2\pi}{5}t\right)}$

Note: both of these functions can be graphed using the Trigonometry app available at www.mathguy.us.

9) An object in simple harmonic motion has a frequency of $\frac{5}{2}$ oscillations per second and an amplitude of 3 feet. Write an equation in the form d = a sin ωt for the object's simple harmonic motion.

Assuming that $d = 0$ at time $t = 0$, it makes sense to use a sine function for this problem. Since the amplitude is 3 feet, a good representation of this would be a sine curve with lead coefficient $= 3$, because $\sin(0) = 0$, whereas, $\cos(0) = 0$.

Recalling that $\omega = 2\pi f$, and with $f = \frac{5}{2}$ we get: $\omega = 2\pi \cdot \frac{5}{2} = 5\pi$.

The resulting equation, then, is: $\mathbf{d = 3\sin(5\pi t)}$

Complete the identity.

10) $\dfrac{\cos^2 x - \sin^2 x}{1 - \tan^2 x} = ?$

A) -1 B) 1 C) $\cos^2 x$ D) $\sin^2 x$

$$\frac{\cos^2 x - \sin^2 x}{1 - \tan^2 x} = \frac{\cos^2 x - \sin^2 x}{\frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}} = \frac{\cos^2 x - \sin^2 x}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}}$$

$$\frac{\cos^2 x - \sin^2 x}{1} \cdot \frac{\cos^2 x}{\cos^2 x - \sin^2 x} = \mathbf{\cos^2 x} \qquad \textbf{Answer C}$$

11) $\sin^2 x + \sin^2 x \cot^2 x = ?$

A) $\cot^2 x + 1$ B) 1 C) $\cot^2 x - 1$ D) $\sin^2 x + 1$

$$\sin^2 x + \sin^2 x \cdot \cot^2 x = \sin^2 x + \sin^2 x \cdot \frac{\cos^2 x}{\sin^2 x} = \sin^2 x + \cos^2 x = \mathbf{1} \qquad \text{Answer B}$$

Use a half-angle formula to find the exact value of the expression.

12) $\sin 75°$

A) $\frac{1}{2}\sqrt{2 - \sqrt{3}}$ B) $-\frac{1}{2}\sqrt{2 + \sqrt{3}}$ C) $\frac{1}{2}\sqrt{2 + \sqrt{3}}$ D) $-\frac{1}{2}\sqrt{2 - \sqrt{3}}$

$$\sin 75° = \sin\left(\frac{1}{2} \cdot 150°\right) = \sqrt{\frac{1 - \cos 150°}{2}} = \sqrt{\frac{1 - \left(-\frac{\sqrt{3}}{2}\right)}{2}} = \frac{\sqrt{2 + \sqrt{3}}}{2} \qquad \textbf{Answer C}$$

Note: the half-angle formula has a " $\pm$ " sign in front. We use " $+$ " for this problem because 75° is in Q1, and sine values are positive in Q1.

13) $\cos \frac{3\pi}{8}$

A) $\frac{1}{2}\sqrt{2 - \sqrt{2}}$ B) $-\frac{1}{2}\sqrt{2 - \sqrt{2}}$ C) $-\frac{1}{2}\sqrt{2 + \sqrt{2}}$ D) $\frac{1}{2}\sqrt{2 + \sqrt{2}}$

$$\cos\frac{3\pi}{8} = \cos\left(\frac{1}{2} \cdot \frac{3\pi}{4}\right) = \sqrt{\frac{1 + \cos\frac{3\pi}{4}}{2}} = \sqrt{\frac{1 + \left(-\frac{\sqrt{2}}{2}\right)}{2}} = \frac{\sqrt{2 - \sqrt{2}}}{2} \qquad \textbf{Answer A}$$

Note: the half-angle formula has a " $\pm$ " sign in front. We use " $+$ " for this problem because $\frac{3\pi}{8}$ is in Q1, and cosine values are positive in Q1.

SHORT ANSWER. Write the word or phrase that best completes each statement or answers the question.

Solve the equation on the interval $[0, 2\pi)$.

14) $\tan^2 x \sin x = \tan^2 x$

$$\tan^2 x \sin x = \tan^2 x$$

$$\tan^2 x \sin x - \tan^2 x = 0$$

$$\tan^2 x (\sin x - 1) = 0$$

$\tan x = 0$ or $(\sin x - 1) = 0$

$x = 0, \pi$

$\sin x = 1$

$x = \frac{\pi}{2}$

While $x = \frac{\pi}{2}$ is a solution to the equation $\sin x = 1$, $\tan x$ is undefined at $x = \frac{\pi}{2}$.

Therefore, the original equation,

$$\tan^2 x \sin x = \tan^2 x$$

is undefined at $x = \frac{\pi}{2}$.

So $x = \frac{\pi}{2}$ is not a solution to this equation.

$$\boldsymbol{x = 0, \pi}$$

Solve the equation on the interval $[0, 2\pi)$.

15) $2\cos^2 x + \sin x - 2 = 0$

$$2\cos^2 x + \sin x - 2 = 0$$

$$2(1 - \sin^2 x) + \sin x - 2 = 0$$

$$2 - 2\sin^2 x + \sin x - 2 = 0$$

$$-2\sin^2 x + \sin x = 0$$

$$\sin x(-2\sin x + 1) = 0$$

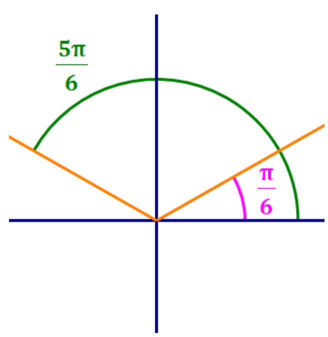
When an equation contains more than one function, try to convert it to one that contains only one function.

$\sin x = 0$ or $(-2\sin x + 1) = 0$

$x = 0, \pi$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$



$$\boldsymbol{x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi}$$

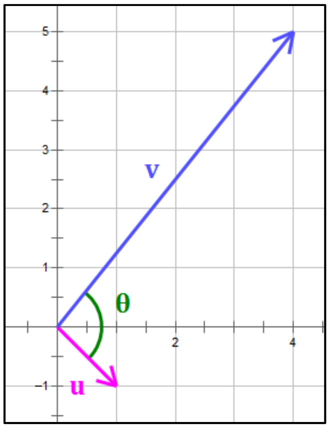
Find the angle between the given vectors. Round to the nearest tenth of a degree.

16) a.) u = i - j, v = 4i + 5j   b.) u = -i + 2j, v = 3i - 3j

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\ \|\mathbf{v}\|} \qquad 0° \le \theta \le 180°$$

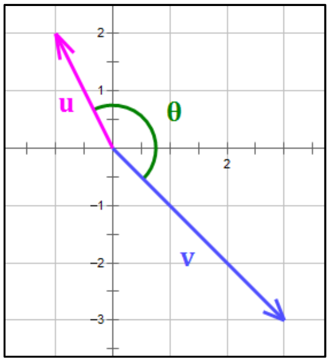
a)

$$\mathbf{u} = \langle 1, -1\rangle$$
$$\cdot\ \mathbf{v} = \langle 4,\ \ 5\rangle$$
$$\mathbf{u}\cdot\mathbf{v} = (1\cdot 4) + ([-1]\cdot 5) = -1$$
$$\|\mathbf{u}\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
$$\|\mathbf{v}\| = \sqrt{4^2 + 5^2} = \sqrt{41}$$
$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\ \|\mathbf{v}\|} = \frac{-1}{\sqrt{2}\cdot\sqrt{41}} = \frac{-1}{\sqrt{82}}$$
$$\boldsymbol{\theta} = \cos^{-1}\left(\frac{-1}{\sqrt{82}}\right) = \mathbf{96.3°}$$



b)

$$\mathbf{u} = \langle -1,\ \ 2\rangle$$
$$\cdot\ \mathbf{v} = \langle\ \ 3, -3\rangle$$
$$\mathbf{u}\cdot\mathbf{v} = (-1\cdot 3) + (2\cdot[-3]) = -9$$
$$\|\mathbf{u}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$
$$\|\mathbf{v}\| = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$
$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\ \|\mathbf{v}\|} = \frac{-9}{\sqrt{5}\cdot 3\sqrt{2}} = \frac{-3}{\sqrt{10}}$$
$$\boldsymbol{\theta} = \cos^{-1}\left(\frac{-3}{\sqrt{10}}\right) = \mathbf{161.6°}$$



Convert the polar equation to a rectangular equation.

17) a.) $r = -3\cos\theta$

Substitute $\cos\theta = \frac{x}{r}$ and $r^2 = x^2 + y^2$

$r = -3\left(\frac{x}{r}\right)$

$r^2 = -3x$

$x^2 + y^2 = -3x$

$x^2 + 3x \qquad + y^2 = 0$

$\left(x^2 + 3x + \frac{9}{4}\right) + y^2 = \frac{9}{4}$

$\left(x + \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

b.) $r\cos\theta = 6$

Substitute $x = r\cos\theta$

$x = 6$

18) $r = 4\cos\theta + 8\sin\theta$

Substitute $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$ and $r^2 = x^2 + y^2$

$r = 4\left(\frac{x}{r}\right) + 8\left(\frac{y}{r}\right)$

$r^2 = 4x + 8y$

$x^2 + y^2 = 4x + 8y$

$x^2 - 4x \qquad + y^2 - 8y \qquad = 0$

$(x^2 - 4x + 4) + (y^2 - 8y + 16) = 4 + 16$

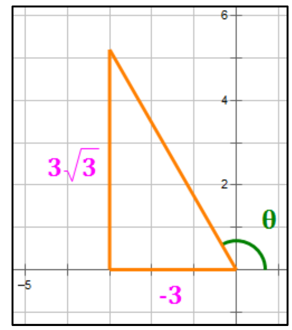
$(x - 2)^2 + (y - 4)^2 = 20$

Write the complex number in polar form. Express the argument in radians.

19) a.) $-3 + 3\sqrt{3}i$ b.) $-5\sqrt{3} - 5i$

a) The process of putting a complex number in polar form is very similar to converting a set of rectangular coordinates to polar coordinates. So, if this process seems familiar, that's because it is.

$$z = -3 + 3\sqrt{3}i$$

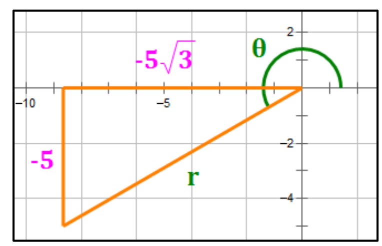


$$r = \sqrt{(-3)^2 + (3\sqrt{3})^2} = 6$$

$$\theta = \tan^{-1}\left(\frac{3\sqrt{3}}{-3}\right) = \tan^{-1}(-\sqrt{3}) \text{ in } Q2 = \frac{2\pi}{3}$$

So, the polar form is: $6\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \mathbf{6\ cis}\,\frac{\mathbf{2\pi}}{\mathbf{3}}$

b) $z = -5\sqrt{3} - 5i$



$$r = \sqrt{(-5\sqrt{3})^2 + (-5)^2} = 10$$

$$\theta = \tan^{-1}\left(\frac{-5}{-5\sqrt{3}}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) \text{ in } Q3 = \frac{7\pi}{6}$$

So, the polar form is: $10\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = \mathbf{10\ cis}\,\frac{\mathbf{7\pi}}{\mathbf{6}}$

Use DeMoivre's Theorem to find the indicated power of the complex number. Write the answer in rectangular form.

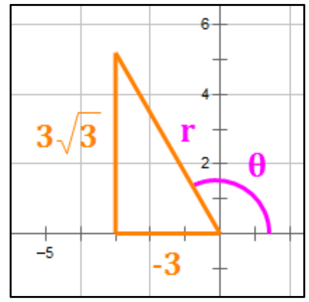
20) a.) $[3(\cos 15° + i \sin 15°)]^4$ b.) $(-3 + 3i\sqrt{3})^3$

$a)$ $[3\ (\cos 15° + i \sin 15°)]^4$

To take a power of two numbers in polar form, take the power of the r-value and multiply the angle by the exponent. (This is the essence of DeMoivre's Theorem.)

$$[3\ (\cos 15° + i \sin 15°)]^4 = 3^4 \cdot \text{cis}(4 \cdot 15°) = 81\ \text{cis}(60°)$$

$$81\ \text{cis}(60°) = 81\ (\cos 60° + i \sin 60°) = 81\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{\mathbf{81}}{\mathbf{2}} + \boldsymbol{i}\frac{\mathbf{81\sqrt{3}}}{\mathbf{2}}$$



$b)\ \left(-3 + 3i\sqrt{3}\right)^3$

$$r = \sqrt{(-3)^2 + \left(3\sqrt{3}\right)^2} = 6$$

$$\theta = \tan^{-1}\left(\frac{3\sqrt{3}}{-3}\right) = \tan^{-1}\left(-\sqrt{3}\right) \text{ in } Q2 = \frac{2\pi}{3}$$

Then, $\left(-3 + 3i\sqrt{3}\right)^3 = \left(6 \text{ cis } \frac{2\pi}{3}\right)^3 = 6^3 \text{ cis } \left(3 \cdot \frac{2\pi}{3}\right)$

$$= 216 \text{ cis } 2\pi = 216 \cdot (\cos 2\pi + i \sin 2\pi) = \mathbf{216}$$