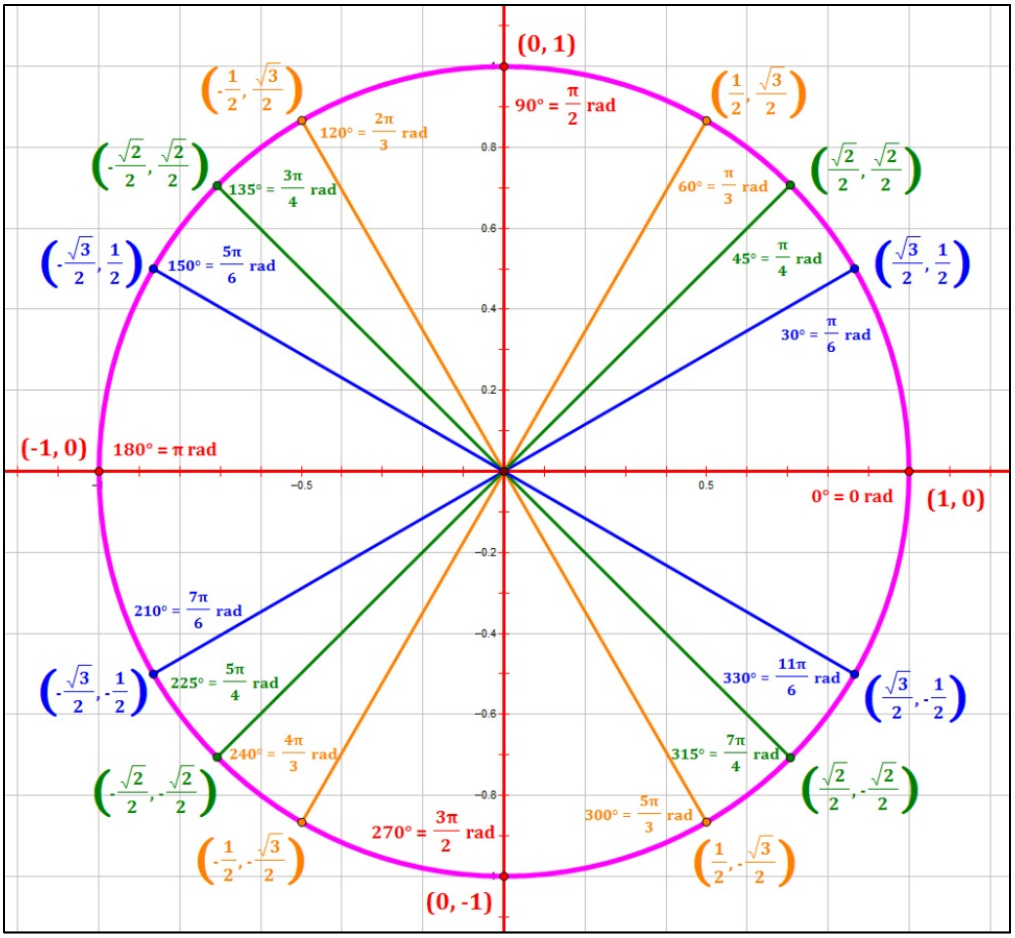
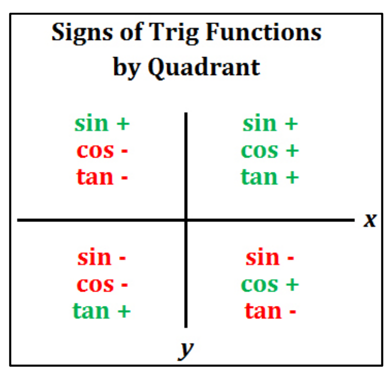
Trigonometry (Chapter 6) – Sample Test #2

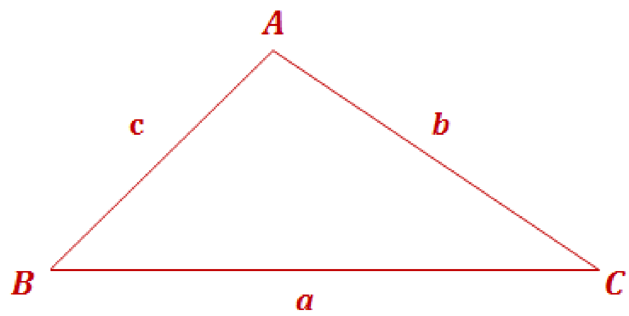
First, a couple of things to help out:



Trig Functions of Special Angles (θ)				
Radians	**Degrees**	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0°	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{\sqrt{4}} = 0$
$\pi/6$	30°	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\pi/4$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{\sqrt{2}} = 1$
$\pi/3$	60°	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{\sqrt{1}} = \sqrt{3}$
$\pi/2$	90°	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{2} = 0$	*undefined*



More Formulas (memorize these):



Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Area of a Triangle:

$$A = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B = \frac{1}{2} bc \sin A$$

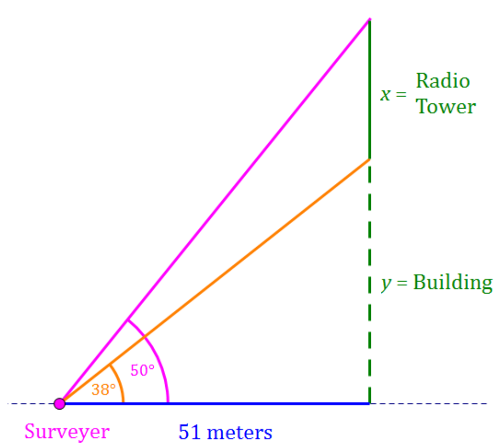
$$A = \sqrt{s(s-a)(s-b)(s-c)} \qquad s = \frac{1}{2}(a+b+c)$$

Find the area of the triangle having the given measurements. Round to the nearest square unit.

1) C = 120°, a = 4 yards, b = 5 yards

$$\textbf{Area} = \frac{1}{2} ab \sin C = \frac{1}{2} \cdot 4 \cdot 5 \cdot \sin 120° = \mathbf{9 \text{ yards}^2}$$

2) A surveyor standing 51 meters from the base of a building measures the angle to the top of the building and finds it to be 38°. The surveyor then measures the angle to the top of the radio tower on the building and finds that it is 50°. How tall is the radio tower?



$$\tan 38° = \frac{y}{51}$$

$$y = 51 \cdot \tan 38° = 39.8456 \text{ meters}$$

$$\tan 50° = \frac{x+y}{51}$$

$$x + y = 51 \cdot \tan 50° = 60.7794 \text{ meters}$$

$$x = 60.7794 - 39.8456 \sim \mathbf{20.93} \text{ meters}$$

Solve the triangle. Round lengths to the nearest tenth and angle measures to the nearest degree.

When you are given the lengths of two sides and the measure of the angle between them:

- Use the Law of Cosines to determine the third side length,
- Use the Law of Sines to determine the measure of one of the two unknown angles, and
- Subtract the two known angle measures from 180° to get the measure of the last unknown angle.

$$\left.\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}\right\}\text{ Law of Cosines}$$

Note: in problems with a lot of calculations, it is a good idea to keep more accuracy than you are required to provide in the final answer, and round your answer at the end. This avoids the compounding of rounding errors throughout your calculations.

3) a = 6, c = 12, B = 124°

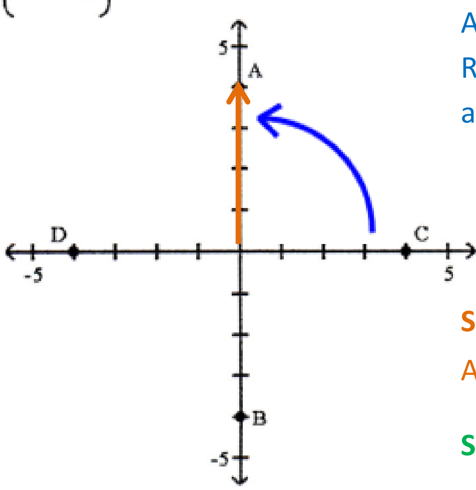
$$\boldsymbol{b} = \sqrt{6^2 + 12^2 - 2(6)(12)(\cos 124°)} \sim 16.14075 \sim \mathbf{16.1}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \Rightarrow \quad \frac{6}{\sin A} = \frac{16.14075}{\sin 124°} \quad \Rightarrow \quad \sin A = 0.3082 \quad \Rightarrow \quad \boldsymbol{m\angle A = 18°}$$

$$\boldsymbol{m\angle C} = 180° - 124° - 18° = \mathbf{38°}$$

Match the point in polar coordinates with either A, B, C, or D on the graph.

4) $\left(-4, -\frac{\pi}{2}\right)$



Step 1:

A negative radius is confusing, let's make it positive. Recall that changing sign of r requires you to either add or subtract π radians (180°) from θ.

$$\left(-4, -\frac{\pi}{2}\right) = \left(4, -\frac{\pi}{2} + \pi\right) = \left(4, \frac{\pi}{2}\right)$$

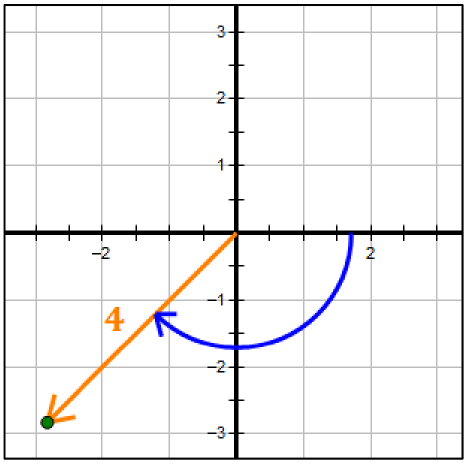
Step 2:

At an angle of $\theta = \frac{\pi}{2}$, move a distance of 4.

Step 3: Answer: **Point A**

Use a polar coordinate system to plot the point with the given polar coordinates.

5) $\left(4, \frac{-3\pi}{4}\right)$



Step 1:

The angle $\left(-\frac{3\pi}{4}\right)$ is at 225°.

Step 2:

The radius (4) means we move in the direction of the angle a distance of 4.

Step 3:

Answer: Shown as a green point on the graph.

Alternatively, convert the polar coordinates (r, θ) to rectangular coordinates as follows:

$$r = 4, \quad \theta = -\frac{3\pi}{4}$$

$$x = r\cos\theta = 4\cos\left(-\frac{3\pi}{4}\right) = 4 \cdot \left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2} \sim -2.8$$

$$y = r\sin\theta = 4\sin\left(-\frac{3\pi}{4}\right) = 4 \cdot \left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2} \sim -2.8$$

So, the rectangular coordinates are: $(\mathbf{-2.8, -2.8})$

Polar coordinates of a point are given. Find the rectangular coordinates of the point. Give exact answer.

Recall that we can change the sign of r, and either add or subtract 180° from θ, and get a new representation for the <u>same</u> point!

6) (-3, 270°)

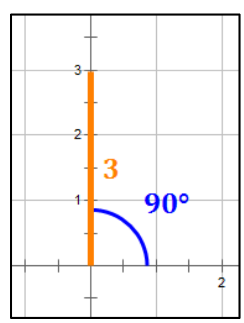
First, note that $(-3, 270°) = (3, 270° - 180°) = (3, 90°)$

$$r = 3, \quad \theta = 90°$$

$$x = r\cos\theta = 3\cos(90°) = 3 \cdot (0) = 0$$

$$y = r\sin\theta = 3\sin(90°) = 3 \cdot (1) = 3$$

So, the rectangular coordinates are: $(\mathbf{0, 3})$



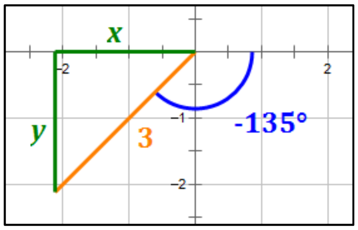
7) (3, -135°)

$r = 3, \quad \theta = -135°$

$x = r\cos\theta = 3\cos(-135°) = 3\cdot\left(-\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{2}$

$y = r\sin\theta = 3\sin(-135°) = 3\cdot\left(-\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{2}$

So, the rectangular coordinates are: $\left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$



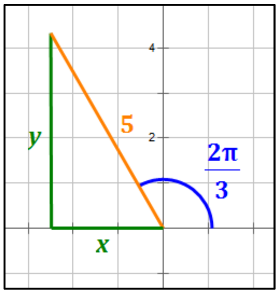
8) $(5, \frac{2\pi}{3})$

$r = 5, \quad \theta = \frac{2\pi}{3}$

$x = r\cos\theta = 5\cos\left(\frac{2\pi}{3}\right) = 5\cdot\left(-\frac{1}{2}\right) = -\frac{5}{2}$

$y = r\sin\theta = 5\sin\left(\frac{2\pi}{3}\right) = 5\cdot\left(\frac{\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{2}$

So, the rectangular coordinates are: $\left(-\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$



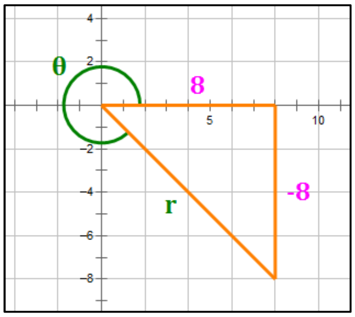
The rectangular coordinates of a point are given. Find polar coordinates of the point. Express θ in radians.

9) (8,-8)

$r = \sqrt{8^2 + (-8)^2} = 8\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-8}{8}\right) = \tan^{-1}(-1) \text{ in } Q4 = \frac{7\pi}{4}$

So, the polar coordinates are: $\left(8\sqrt{2}, \frac{7\pi}{4}\right)$

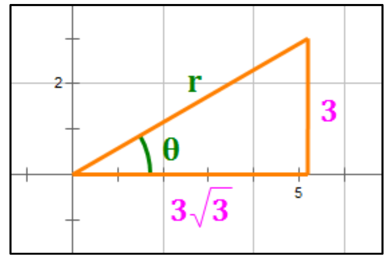


10) $(3\sqrt{3}, 3)$

$$r = \sqrt{(3\sqrt{3})^2 + 3^2} = 6$$

$$\theta = \tan^{-1}\left(\frac{3}{3\sqrt{3}}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) \text{ in } Q1 = \frac{\pi}{6}$$

So, the polar coordinates are: $\left(\mathbf{6}, \frac{\pi}{6}\right)$

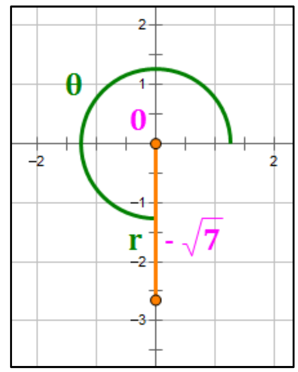


11) $(0, -\sqrt{7})$

$$r = \sqrt{0^2 + (-\sqrt{7})^2} = \sqrt{7}$$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{7}}{0}\right) = \text{undefined at: } \theta = \frac{3\pi}{2}$$

So, the polar coordinates are: $\left(\sqrt{7}, \frac{3\pi}{2}\right)$ or $\left(-\sqrt{7}, \frac{\pi}{2}\right)$



Convert the rectangular equation to a polar equation that expresses r in terms of θ.

12) $y = 1$

Since $y = r\sin\theta$, we make that substitution and solve for r.

$$r\sin\theta = 1$$

$$\boldsymbol{r = \frac{1}{\sin\theta}} \quad \text{or} \quad \boldsymbol{r = \csc\theta}$$

13) $x^2 + y^2 = 25$

Substitute $x = r\cos\theta$ and $y = r\sin\theta$.

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 25$$

$$r^2(\cos^2\theta + \sin^2\theta) = 25$$ (recall that: $\cos^2\theta + \sin^2\theta = 1$)

$$r^2 = 25$$

$$\boldsymbol{r = 5}$$ (note that we take the positive root of r only)

Convert the polar equation to a rectangular equation.

14) r = 3

Substitute $r = \sqrt{x^2 + y^2}$

$\sqrt{x^2 + y^2} = 3$

$\boldsymbol{x^2 + y^2 = 9}$

15) r = 6 csc θ

$r = \dfrac{6}{\sin\theta}$

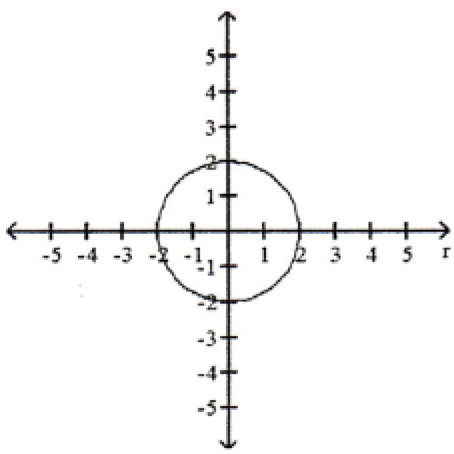
$r \sin\theta = 6$

Substitute $y = r\sin\theta$

$\boldsymbol{y = 6}$

The graph of a polar equation is given. Select the polar equation for the graph.

16)



The graph has a constant radius, no matter what the value of the angle. It is also obviously a circle of radius 2. The polar equation, then, is:

$\boldsymbol{r = 2}$

See how much simpler this is than the rectangular equation:

$x^2 + y^2 = 4$

Find the absolute value of the complex number.

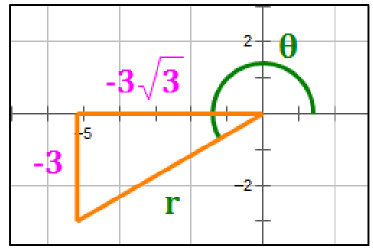
17) $z = -8 + 2i$

$r = \sqrt{(-8)^2 + 2^2} = \sqrt{68} = \boldsymbol{2\sqrt{17}}$

Write the complex number in polar form. Express the argument in radians.

18) $z = -3\sqrt{3} - 3i$

The process of putting a complex number in polar form is very similar to converting a set of rectangular coordinates to polar coordinates. So, if this process seems familiar, that's because it is.



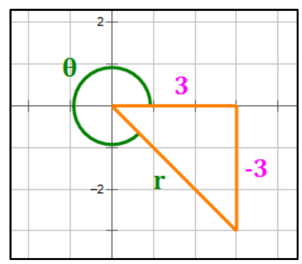
$r = \sqrt{\left(-3\sqrt{3}\right)^2 + (-3)^2} = 6$

$\theta = \tan^{-1}\left(\frac{-3}{-3\sqrt{3}}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) \text{ in } Q3 = \frac{7\pi}{6}$

So, the polar form is: $\mathbf{6\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 6 \text{ cis}\frac{7\pi}{6}}$

Note: $6 \text{ cis}\frac{7\pi}{6} = 6e^{\left(i\cdot\frac{7\pi}{6}\right)}$ because $e^{i\theta} = \cos\theta + i\sin\theta$. However, the student may not be required to know this at this point in the course.

19) $z = 3 - 3i$



$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$

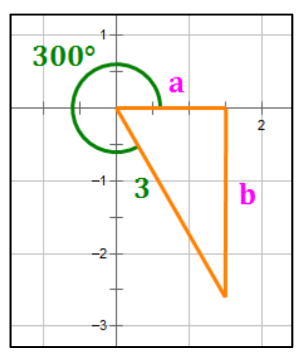
$\theta = \tan^{-1}\left(\frac{-3}{3}\right) = \tan^{-1}(-1) \text{ in } Q4 = \frac{7\pi}{4}$

So, the polar form is: $\mathbf{3\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 3\sqrt{2} \text{ cis}\frac{7\pi}{4}}$

Note: $3\sqrt{2} \text{ cis}\frac{7\pi}{4} = 3\sqrt{2}\, e^{\left(i\cdot\frac{7\pi}{4}\right)}$ because $e^{i\theta} = \cos\theta + i\sin\theta$. However, the student may not be required to know this at this point in the course.

Write the complex number in rectangular form. Give exact answer.

20) $z = -3(\cos 120° + i\sin 120°)$



$z = -3 \text{ cis } 120° = 3 \text{ cis } (120° + 180°) = 3 \text{ cis } 300°$

We want this in the form $a + bi$

$a = r\cos\theta = 3\cos 300° = 3\cdot\left(\frac{1}{2}\right) = \frac{3}{2}$

$b = r\sin\theta = 3\sin 300° = 3\cdot\left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2}$

So, the rectangular form is: $\mathbf{\frac{3}{2} - i\,\frac{3\sqrt{3}}{2}}$

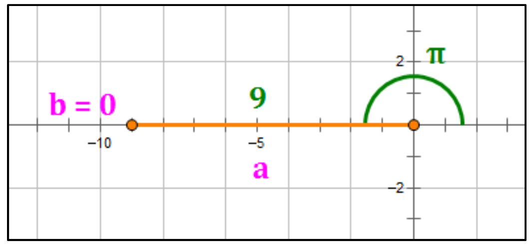
21) $9(\cos\pi + i\sin\pi)$

We want this in the form $a + bi$

$a = r\cos\theta = 9\cos\pi = 9\cdot(-1) = -9$

$b = r\sin\theta = 9\sin\pi = 9\cdot(0) = 0$

So, the rectangular form is: $\mathbf{-9}$



Find the product of the complex numbers. Leave answer in polar form.

22) $z_1 = \sqrt{3}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$ ⟵ shorthand is: $z_1 = \sqrt{3}\,\text{cis}\left(\frac{7\pi}{4}\right)$

$z_2 = \sqrt{6}\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)$ ⟵ shorthand is: $z_2 = \sqrt{6}\,\text{cis}\left(\frac{9\pi}{4}\right)$

To multiply two numbers in polar form, multiply the r-values and add the angles.

$$z_1\cdot z_2 = \sqrt{3}\cdot\sqrt{6}\cdot\text{cis}\left(\frac{7\pi}{4}+\frac{9\pi}{4}\right) = 3\sqrt{2}\,\text{cis}(4\pi) = \mathbf{3\sqrt{2}\,cis\,0}$$

Note: multiplication may be easier to understand in exponential form, since exponents are added when values with the same base are multiplied:

$$\sqrt{3}e^{i\cdot\frac{7\pi}{4}}\cdot\sqrt{6}e^{i\cdot\frac{9\pi}{4}} = \sqrt{3}\cdot\sqrt{6}\cdot e^{i\left(\frac{7\pi}{4}+\frac{9\pi}{4}\right)} = 3\sqrt{2}\,e^{i(4\pi)} = 3\sqrt{2}\,e^{i(0)} = 3\sqrt{2}$$

23) $z_1 = 4i$

$z_2 = -6 + 6i$

This problem is probably best approached by converting each number to polar form and then multiplying.

Relating to z_1:

$$r = \sqrt{0^2 + (4)^2} = 4 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{4}{0}\right) = \text{undefined at: } \theta = \frac{\pi}{2}$$

So, the polar form is: $4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 4\,\text{cis}\left(\frac{\pi}{2}\right)$

Relating to z_2:

$$s = \sqrt{(-6)^2 + 6^2} = 6\sqrt{2} \quad \text{and} \quad \varphi = \tan^{-1}\left(\frac{6}{-6}\right) = \tan^{-1}(-1) \text{ in } Q2 = \frac{3\pi}{4}$$

So, the polar form is: $6\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 6\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$

Then, multiply:

$$z_1\cdot z_2 = \left(4\,\text{cis}\,\frac{\pi}{2}\right)\cdot\left(6\sqrt{2}\,\text{cis}\,\frac{3\pi}{4}\right) = 24\sqrt{2}\,\text{cis}\left(\frac{\pi}{2}+\frac{3\pi}{4}\right) = \mathbf{24\sqrt{2}\,cis}\left(\frac{5\pi}{4}\right)$$

Find the quotient $\frac{z_1}{z_2}$ of the complex numbers. Leave answer in polar form.

24) $z_1 = \sqrt{3}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

$z_2 = \sqrt{6}\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)$

To divide two numbers in polar form, divide the r-values and subtract the angles.

$$z_1 \cdot z_2 = \frac{\sqrt{3}}{\sqrt{6}} \cdot \text{cis}\left(\frac{7\pi}{4} - \frac{9\pi}{4}\right) = \frac{1}{\sqrt{2}}\ \text{cis}\left(-\frac{\pi}{2}\right) = \frac{\sqrt{2}}{2}\text{cis}\left(-\frac{\pi}{2} + 2\pi\right) = \mathbf{\frac{\sqrt{2}}{2}\,cis\left(\frac{3\pi}{2}\right)}$$

Note: division may be easier to understand in exponential form, since exponents are subtracted when values with the same base are divided:

$$\frac{\sqrt{3}\,e^{i\cdot\frac{7\pi}{4}}}{\sqrt{6}\,e^{i\cdot\frac{9\pi}{4}}} = \frac{1}{\sqrt{2}}\,e^{i\left(\frac{7\pi}{4}-\frac{9\pi}{4}\right)} = \frac{\sqrt{2}}{2}e^{i\left(-\frac{\pi}{2}\right)} = \frac{\sqrt{2}}{2}e^{i\left(-\frac{\pi}{2}+2\pi\right)} = \frac{\sqrt{2}}{2}e^{i\left(\frac{3\pi}{2}\right)}$$

Use DeMoivre's Theorem to find the indicated power of the complex number. Write the answer in rectangular form.

25) $[3\,(\cos 15° + i\sin 15°)]^4$

To take a power of two numbers in polar form, take the power of the r-value and multiply the angle by the exponent. (This is the essence of DeMoivre's Theorem.)

$$[3\,(\cos 15° + i\sin 15°)]^4 = 3^4 \cdot \text{cis}(4\cdot 15°) = 81\ \text{cis}(60°)$$

$$= 81\,(\cos 60° + i\sin 60°) = 81\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \mathbf{\frac{81}{2} + i\frac{81\sqrt{3}}{2}}$$

Note: Consider this in exponential form. Exponents are multiplied when a value with an exponent is taken to a power:

$$\left(3\ \text{cis}\ \frac{\pi}{12}\right)^4 = \left(3e^{i\cdot\frac{\pi}{12}}\right)^4 = 3^4 e^{4\cdot\left(i\cdot\frac{\pi}{12}\right)} = 81\ e^{\left(i\cdot\frac{\pi}{3}\right)}$$

26) $(-\sqrt{3} + i)^6$

$$r = \sqrt{\left(-\sqrt{3}\right)^2 + (1)^2} = 2 \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)\ \text{in}\ Q2 = \frac{5\pi}{6}$$

Then, $(-\sqrt{3} + i)^6 = \left(2\ \text{cis}\ \frac{5\pi}{6}\right)^6 = 2^6\ \text{cis}\ \left(6\cdot\frac{5\pi}{6}\right)$

$$= 64\ \text{cis}\ 5\pi = 64\cdot(\cos 5\pi + i\sin 5\pi) = \mathbf{-64}$$

Find all the complex roots. Write the answer in the indicated form.

27) The complex cube roots of 8 (rectangular form).

We must use DeMoivre's Theorem for roots to solve this problem.

Let $z = r(\cos\theta + i\sin\theta)$. Then, z has n distinct complex n-th roots that occupy positions equidistant from each other on a circle of radius $\sqrt[n]{r}$. Let's call the roots: $z_1, z_2, \dots, z_n$
Then, these roots can be calculated as follows:

$$z_k = \sqrt[n]{r} \cdot \left[\cos\left(\frac{\theta + k(2\pi)}{n}\right) + i\sin\left(\frac{\theta + k(2\pi)}{n}\right)\right]$$

Given this, let's find the cube roots of $\mathbf{8}$.

First, since $z = a + bi$, we have $a = 8$ and $b = 0$.

Then, $r = \sqrt{8^2 + 0^2} = 8;\ \sqrt[3]{r} = 2$

And, $\theta = \tan^{-1}\left(\frac{0}{8}\right) = 0°\,;\ \frac{\theta}{3} = 0°$

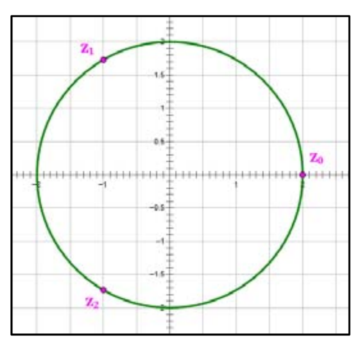
The incremental angle for successive roots is: $360° \div 3 \text{ roots} = 120°$.

Then create a root development chart like this (answers are in green):

Cube roots of 8: $\sqrt[n]{r} = \sqrt[3]{8} = 2$ $\theta_0 = 0°$ incremental $\theta = 120°$		
k	**Angle (θ_k)**	$z_k = \sqrt[n]{r} \cdot \cos\theta_k + \sqrt[n]{r} \cdot \sin\theta_k \cdot i$
0	$0°$	$z_0 = \mathbf{2}$
1	$0° + 120° = 120°$	$z_1 = \mathbf{-1 + \sqrt{3}\,i}$
2	$120° + 120° = 240°$	$z_2 = \mathbf{-1 - \sqrt{3}\,i}$

Notice that if we add another $120°$, we get $360°$, which is equivalent to our first angle, $0°$ because $360° - 360° = 0°$. This is a good thing to check. The "next angle" will always be equivalent to the first angle! If it isn't, go back and check your work.

Roots fit on a circle: Notice that, since all of the roots of $\mathbf{8}$ have the same magnitude, and their angles that are $120°$ apart from each other, that they occupy equidistant positions on a circle with center $(0, 0)$ and radius $\sqrt[3]{8} = 2$.



More information about using DeMoivre's Theorem for Roots, including a more complex example, can be found in the Trigonometry Handbook on www.mathguy.com.

Let v be the vector from initial point P_1 to terminal point P_2. Write v in terms of i and j.

An alternative notation for a vector in the form $a\mathbf{i} + b\mathbf{j}$ is $\langle a, b\rangle$. Using this alternative notation makes many vector operations much easier to work with.

28) $P_1 = (5, 5)$; $P_2 = (-1, 5)$

We can calculate $\mathbf{v}$ as the difference between the two given points.

$$\begin{array}{rl} (-1, 5) = & P_2 \\ -(\ \ 5, 5) = & P_1 \\ \hline \mathbf{v} = \langle -6, 0\rangle & \end{array}$$

So, in terms of $\mathbf{i}$ and $\mathbf{j}$, $\mathbf{v} = -6\mathbf{i}$

Find the specified vector or scalar.

29) **u** = -2**i** - 7**j** and **v** = -4**i** - 21**j**; Find $\|\mathbf{v} - \mathbf{u}\|$.

$$\begin{array}{rl} \mathbf{v} = & \langle -4, -21\rangle \\ +\ \ -\mathbf{u} = & \langle \ \ 2, \ \ \ 7\rangle \\ \hline \mathbf{v} - \mathbf{u} = & \langle -2, -14\rangle \end{array}$$

Subtracting $\mathbf{u}$ is the same as adding $-\mathbf{u}$. To get $-\mathbf{u}$, simply change the sign of each element of $\mathbf{u}$. If you find it easier to add than to subtract, you may want to adopt this approach to subtracting vectors.

$$\|\mathbf{v} - \mathbf{u}\| = \sqrt{(-2)^2 + (-14)^2}$$

$$= \sqrt{200} = \sqrt{100} \cdot \sqrt{2} = \mathbf{10\sqrt{2}}$$

30) **u** = -12**i** - 2**j**, **v** = 6**i** + 7**j**; Find **u** - **v**.

To add or subtract vectors, simply line them up vertically and perform the required operation:

$$\begin{array}{rl} \mathbf{u} = & \langle -12, -2\rangle \\ +\ \ -\mathbf{v} = & \langle \ -6, -7\rangle \\ \hline \mathbf{u} - \mathbf{v} = & \langle -12 - 6, -2 - 7\rangle \end{array}$$

$$\mathbf{u} - \mathbf{v} = \langle -18, -9\rangle = -18\mathbf{i} - 9\mathbf{j}$$

Find the unit vector that has the same direction as the vector v.

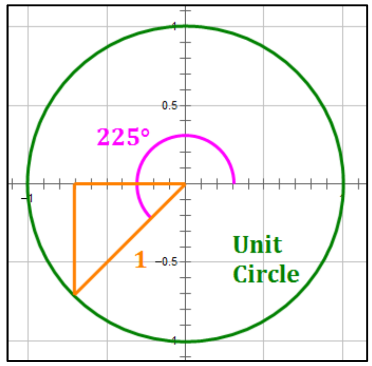
A unit vector has **magnitude 1**. To get a unit vector in the same direction as the original vector, divide the vector by its magnitude.

31) **v** = 3**i** - 4**j**

The unit vector is: $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{3\mathbf{i} - 4\mathbf{j}}{\sqrt{3^2 + (-4)^2}} = \dfrac{3\mathbf{i} - 4\mathbf{j}}{5} = \dfrac{3}{5}\mathbf{i} - \dfrac{4}{5}\mathbf{j}$

Write the vector v in terms of i and j whose magnitude v and direction angle θ are given. Give exact answer.

32) $\|\mathbf{v}\| = 7$, θ = 225°



The unit vector in the direction $\theta = 225°$ is:

$$\langle \cos 225°, \sin 225° \rangle = \langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

Multiply this by $\|\mathbf{v}\| = \mathbf{7}$ to get $\mathbf{v}$:

$$\mathbf{v} = 7\left(-\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}\right) = -\frac{\mathbf{7\sqrt{2}}}{\mathbf{2}}\mathbf{i} - \frac{\mathbf{7\sqrt{2}}}{\mathbf{2}}\mathbf{j}$$

Solve the problem.

33) The magnitude and direction of two forces acting on an object are 60 pounds, N40°E, and 70 pounds, N40°W, respectively. Find the magnitude, to the nearest hundredth of a pound, and the direction angle, to the nearest tenth of a degree, of the resultant force.

This problem requires the addition of two vectors. The approach I prefer is:

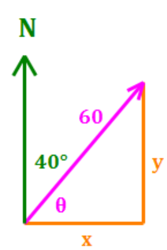
1) Convert each vector into its **i** and **j** components, call them x and y,
2) Add the resulting x and y values for the two vectors, and
3) Convert the sum to its polar form.

Keep additional accuracy throughout and round at the end. This will prevent error compounding and will preserve the required accuracy of your final solutions.

Step 1: Convert each vector into its **i** and **j** components

Let $\mathbf{F_1}$ be a force of 60 lbs. at bearing: N40°E

From the diagram at right,



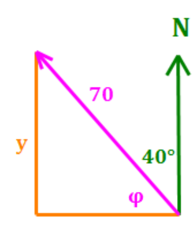
$\theta = 90° - 40° = 50°$

$x = 60 \cos 50° = 38.5673$

$y = 60 \sin 50° = 45.9627$

Let $\mathbf{F_2}$ be a force of 70 lbs. at bearing: N40°W

From the diagram at right,



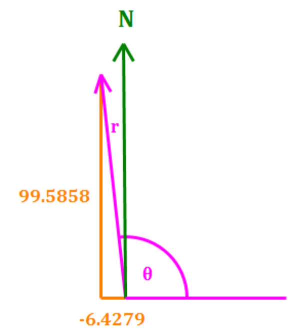
$\varphi = 90° - 40° = 50°$

$x = -70 \cos(50°) = -44.9951$

$y = 70 \sin(50°) = 53.6231$

Step 2: Add the results for the two vectors

$$\mathbf{F_1} = \langle\ \ 38.5673, 45.9627\rangle$$
$$\mathbf{F_2} = \langle -44.9951, 53.6231\rangle$$
$$\mathbf{F_1} + \mathbf{F_2} = \langle\ \ -6.4278, 99.5858\rangle$$



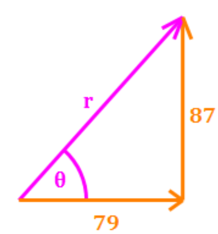
Step 3: Convert the sum to its polar form

Direction Angle $= \boldsymbol{\theta} = \tan^{-1}\left(\frac{99.5858}{-6.4278}\right) = \mathbf{93.7°}$

Magnitude $= \boldsymbol{r} = \sqrt{(-6.4278)^{\mathbf{2}} + 99.5858^{\mathbf{2}}} = \mathbf{99.79}$ **lbs.**

34) One rope pulls a barge directly east with a force of 79 newtons, and another rope pulls the barge directly north with a force of 87 newtons. Find the magnitude of the resultant force acting on the barge.

The process of adding two vectors whose headings are north, east, west or south (NEWS) is very similar to converting a set of rectangular coordinates to polar coordinates. So, if this process seems familiar, that's because it is.



$$r = \sqrt{(79)^2 + (87)^2} = \mathbf{117.52\ newtons}$$

This problem does not require it, but let's calculate the direction angle anyway.

$$\theta = \tan^{-1}\left(\frac{87}{79}\right) = 47.8°$$

Use the given vectors to find the specified scalar.

35) **u** = -5**i** + 3**j**, **v** = 5**i** - 6**j**,**w** = -3**i** + 12**j**; Find **u** · **w** + **v** · **w**.

The alternate notation for vectors comes in especially handy in doing these types of problems. Also, note that: (**u** · **w**) + (**v** · **w**) = (**u** + **v**) · **w**. Let's calculate (**u** + **v**) · **w**.

$$\mathbf{u} = \langle -5,\ \ 3\rangle$$
$$+\ \ \mathbf{v} = \langle\ \ 5, -6\rangle$$
$$\mathbf{u} + \mathbf{v} = \langle\ \ 0, -3\rangle$$
$$\cdot\ \mathbf{w} = \langle -3,\ \ 12\rangle$$
$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (0 \cdot [-3]) + (-3 \cdot 12) = 0 - 36 = \mathbf{-36}$$

Using the distributive property for dot products results in an easier problem with fewer calculations.

36) **u** = -4**i** + 4**j**, **v** = -11**i** - 6**j**; Find **u** · **v**.

The alternate notation for vectors comes in especially handy in calculating a dot product.

$$\mathbf{u} = \langle -4, \ 4\rangle$$
$$\cdot \ \mathbf{v} = \langle -11, -6\rangle$$
$$(\mathbf{u} \cdot \mathbf{v}) = ([-4] \cdot [-11]) + (4 \cdot [-6]) = 44 - 24 = \mathbf{20}$$

Find the angle between the given vectors. Round to the nearest tenth of a degree.

37) **u** = **i** - **j**, **v** = 4**i** + 5**j**

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \ \|\mathbf{v}\|}$$

$$0° \le \theta \le 180°$$

$$\mathbf{u} = \langle 1, -1\rangle$$
$$\cdot \ \mathbf{v} = \langle 4, \ 5\rangle$$
$$\mathbf{u} \cdot \mathbf{v} = (1 \cdot 4) + ([-1] \cdot 5) = -1$$

$$\|\mathbf{u}\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\|\mathbf{v}\| = \sqrt{4^2 + 5^2} = \sqrt{41}$$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \ \|\mathbf{v}\|} = \frac{-1}{\sqrt{2} \cdot \sqrt{41}} = \frac{-1}{\sqrt{82}}$$

$$\boldsymbol{\theta} = \cos^{-1}\left(\frac{-1}{\sqrt{82}}\right) = \mathbf{96.3°}$$

Use the dot product to determine whether the vectors are parallel, orthogonal, or neither.

If the vectors are parallel, one is a multiple of the other; also $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \ \|\mathbf{w}\|$.

If the vectors are perpendicular, their dot product is zero.

38) **v** = 3**i** + 4**j**, **w** = 6**i** + 8**j**

$$\mathbf{v} = \langle 3, 4\rangle$$
$$\mathbf{w} = \langle 6, 8\rangle$$

Clearly, $\mathbf{w} = 2\mathbf{v}$

Therefore, **the vectors are parallel.**

It is clearly easier to check whether one vector is a multiple of the other than to use the dot product method. The student may use either, unless instructed to use a particular method.

To determine if two vectors are parallel using the dot product, we check to see if:

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \ \|\mathbf{w}\|$$

$$\mathbf{v} = \langle 3, 4\rangle$$
$$\cdot \ \mathbf{w} = \langle 6, 8\rangle$$
$$\mathbf{v} \cdot \mathbf{w} = 18 + 32 = 50$$

$$\|\mathbf{v}\| = \sqrt{(3)^2 + (4)^2} = 5$$

$$\|\mathbf{w}\| = \sqrt{(6)^2 + (8)^2} = 10$$

$$\|\mathbf{v}\| \ \|\mathbf{w}\| = 5 \cdot 10 = 50 = \mathbf{v} \cdot \mathbf{w}$$

Therefore, **the vectors are parallel.**

39) **v** = 4**i** + 3**j**, **w** = 3**i** - 4**j**

Calculate the dot product.

$$\begin{array}{r} \mathbf{v} = \langle 4, \quad 3 \rangle \\ \cdot\ \mathbf{w} = \langle 3, -4 \rangle \\ \hline \end{array}$$

$$\mathbf{v} \cdot \mathbf{w} = (4 \cdot 3) + (3 \cdot [-4]) = 12 + (-12) = 0$$

Therefore, **the vectors are orthogonal**.