Trigonometry – Chapter 5 Review

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Complete the identity.

1) $\sec x - \dfrac{1}{\sec x} = ?$ 1) ______

A) $\sec x \csc x$ B) $-2\tan^2 x$ C) $1 + \cot x$ D) $\sin x \tan x$

$$\sec x - \frac{1}{\sec x} = \frac{1}{\cos x} - \cos x$$

$$= \frac{1}{\cos x} - \cos x \cdot \frac{\cos x}{\cos x} = \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}$$

$$= \frac{1-\cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x} = \sin x \cdot \frac{\sin x}{\cos x}$$

$$= \sin x \cdot \tan x$$

Answer D

2) $\tan(\pi - \theta) = ?$ 2) ______

A) $-\tan\theta$ B) $\cot\theta$ C) $-\cot\theta$ D) $\tan\theta$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\,\tan\beta}$$

$$\tan(\pi - \theta) = \frac{\tan\pi - \tan\theta}{1 + \tan\pi\,\tan\theta} = \frac{0 - \tan\theta}{1 + 0\cdot(\tan\theta)} = -\tan\theta$$

Answer A

SHORT ANSWER. Write the word or phrase that best completes each statement or answers the question.

Use a half-angle formula to find the exact value of the expression.

3) $\cos\dfrac{5\pi}{12}$ 3) ______________

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Note that $\dfrac{5\pi}{12}$ is in Quadrant 1, so: $\cos\dfrac{5\pi}{12} > 0$

$$\cos\frac{5\pi}{12} = \sqrt{\frac{1 + \cos\frac{5\pi}{6}}{2}} = \sqrt{\frac{1 + \frac{-\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

Note: $\dfrac{\sqrt{2-\sqrt{3}}}{2} = \dfrac{\sqrt{\frac{4-2\sqrt{3}}{2}}}{2} = \dfrac{\sqrt{3-2\sqrt{3}+1}}{2\sqrt{2}} = \dfrac{\sqrt{(\sqrt{3}-1)^2}}{2\sqrt{2}} = \dfrac{\sqrt{3}-1}{2\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$, which is also correct.

Use trigonometric identities to find the exact value.

4) $\dfrac{\tan 40° + \tan 110°}{1 - \tan 40° \tan 110°}$ 4) ____________

$$\tan\ (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha\ \tan \beta}$$

$$\frac{\tan 40° + \tan 110°}{1 - \tan 40°\ \tan 110°} = \tan\ (40° + 110°) = \tan 150° = -\frac{\sqrt{3}}{3}$$

Solve the equation on the interval $[0, 2\pi)$.

5) $2 \sin^2 x = \sin x$ 5) ____________

$$2\sin^2 x = \sin x$$

$$2\sin^2 x - \sin x = 0$$

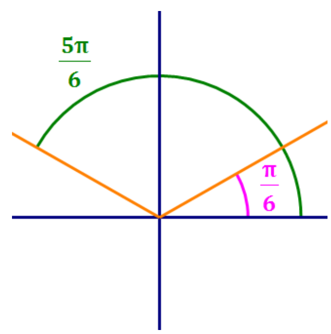
$$\sin x\ (2\sin x - 1) = 0$$

$\sin x = 0$ or $(2\sin x - 1) = 0$

$x = 0, \pi$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$



$$x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$$

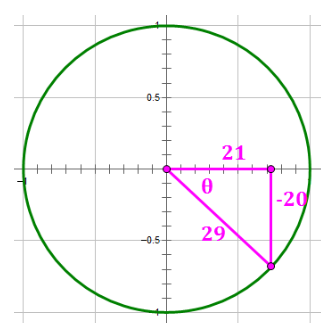
Use the given information to find the exact value of the expression.

6) $\cos \theta = \frac{21}{29}$, θ lies in quadrant IV Find $\sin 2\theta$. 6) ____________

$$\sin 2\theta = 2 \cdot \sin\theta \cdot \cos\theta$$

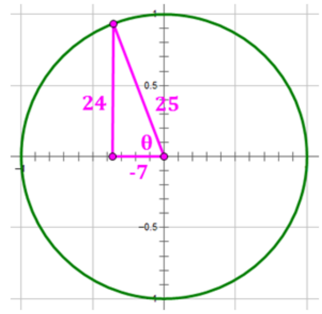
$$\sin 2\theta = 2 \cdot \frac{-20}{29} \cdot \frac{21}{29}$$

$$= \frac{-840}{841}$$



7) $\sin\theta = \frac{24}{25}$, θ lies in quadrant II Find $\tan 2\theta$. 7) ____________

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\cdot\left(-\frac{24}{7}\right)}{1-\left(-\frac{24}{7}\right)^2} = \frac{-\frac{48}{7}}{\frac{-527}{49}} = \mathbf{\frac{336}{527}}$$

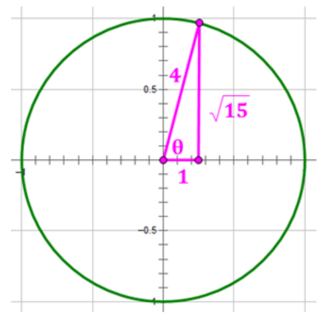


Use the given information to find the exact value of the trigonometric function.

8) $\sec\theta = 4$, θ lies in quadrant I Find $\cos\frac{\theta}{2}$. 8) ____________

$$\sec\theta = 4$$

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{4}$$



Note that θ is in Quadrant 1, so $\frac{\theta}{2}$ is in Quadrant 1.

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\frac{1}{4}}{2}} = \sqrt{\frac{\frac{5}{4}}{2}} = \sqrt{\frac{5}{8}}\cdot\sqrt{\frac{2}{2}} = \sqrt{\frac{10}{16}} = \mathbf{\frac{\sqrt{10}}{4}}$$

Verify the identity.

9) $\csc^2 u - \cos u \sec u = \cot^2 u$ 9) ____________

$$\csc^2 u - \cos u\sec u \qquad = \cot^2 u$$

$$\frac{1}{\sin^2 u} - \cos u\cdot\frac{1}{\cos u}$$

$$\frac{1}{\sin^2 u} - 1$$

$$\frac{1}{\sin^2 u} - \frac{\sin^2 u}{\sin^2 u}$$

$$\frac{\cos^2 u}{\sin^2 u}$$

$$\mathbf{\cot^2 u} \qquad = \mathbf{\cot^2 u}$$

10) $\sin(\alpha - \beta)\cos(\alpha + \beta) = \sin\alpha\cos\alpha - \sin\beta\cos\beta$ 10) ____________

$\sin(\alpha-\beta)\cos(\alpha+\beta) \qquad = \sin\alpha\cos\alpha - \sin\beta\cos\beta$

$(\sin\alpha\cos\beta - \cos\alpha\sin\beta)(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$

$\sin\alpha\cos\beta\cos\alpha\cos\beta - \sin\alpha\cos\beta\sin\alpha\sin\beta - \cos\alpha\sin\beta\cos\alpha\cos\beta$
$+ \cos\alpha\sin\beta\sin\alpha\sin\beta$

$\sin\alpha\cos\alpha\cos^2\beta - \sin^2\alpha\sin\beta\cos\beta - \cos^2\alpha\sin\beta\cos\beta + \sin\alpha\cos\alpha\sin^2\beta$

$(\sin\alpha\cos\alpha\cos^2\beta + \sin\alpha\cos\alpha\sin^2\beta) - (\sin^2\alpha\sin\beta\cos\beta + \cos^2\alpha\sin\beta\cos\beta)$

$\sin\alpha\cos\alpha(\cos^2\beta + \sin^2\beta) - (\sin^2\alpha + \cos^2\alpha)\sin\beta\cos\beta$

$\sin\alpha\cos\alpha(1) - (1)\sin\beta\cos\beta$

$\boldsymbol{\sin\alpha\cos\alpha - \sin\beta\cos\beta \qquad = \sin\alpha\cos\alpha - \sin\beta\cos\beta}$

11) $\tan\theta \cdot \csc\theta = \sec\theta$ 11) ____________

$\tan\theta \cdot \csc\theta \qquad = \sec\theta$

$\frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\sin\theta}$

$\frac{1}{\cos\theta}$

$\boldsymbol{\sec\theta \qquad = \sec\theta}$

12) $\cos 4\theta = 2\cos^2(2\theta) - 1$ (Start with left-hand side) 12) ____________

Note that $4\theta = 2(2\theta)$

Use the formula: $\cos 2\alpha = 2\cos^2\alpha - 1$. Let $\alpha = 2\theta$. Then,

$\cos 4\theta \qquad = 2\cos^2(2\theta) - 1$

$\boldsymbol{2\cos^2(2\theta) - 1 \qquad = 2\cos^2(2\theta) - 1}$

13) $\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$ 13) ______________

Use the formula: $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$\sin\left(\frac{3\pi}{2} - \theta\right) \qquad\qquad = -\cos\theta$$

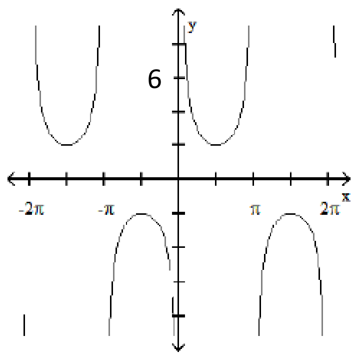
$$\sin\left(\frac{3\pi}{2}\right)\cos\theta - \cos\left(\frac{3\pi}{2}\right)\sin\theta$$

$$(-1)\cos\theta - (0)\sin\theta$$

$$\boldsymbol{-\cos\theta} \qquad\qquad \boldsymbol{= -\cos\theta}$$

Use the graph to complete the identity.

14) $\frac{1 + \cos x}{\sin x} + \frac{\sin x}{1 + \cos x} = ?$ 14) ______________



Let's take a look at the graph. The "loopy-ness" of the graph indicates it is either the secant or cosecant function. Since the vertical asymptotes are at integral multiples of π (e.g., $-\pi, 0, \pi, \ldots$), this indicates the cosecant function.

Next, the graph's maxima are at $y = 2$ and minima are at $y = -2$. So, we would expect this graph to be: $y = 2\csc x$.

Verification:

$$\frac{1 + \cos x}{\sin x} + \frac{\sin x}{1 + \cos x} \qquad\qquad = \boldsymbol{2\csc x}$$

$$\frac{1 + \cos x}{1 + \cos x} \cdot \frac{1 + \cos x}{\sin x} + \frac{\sin x}{1 + \cos x} \cdot \frac{\sin x}{\sin x}$$

$$\frac{(1 + \cos x)^2 + \sin^2 x}{(1 + \cos x)\sin x}$$

$$\frac{1 + 2\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)\sin x}$$

$$\frac{2 + 2\cos x}{(1 + \cos x)\sin x}$$

$$\frac{2\,(1 + \cos x)}{(1 + \cos x)\sin x}$$

$$\frac{2}{\sin x}$$

$$\boldsymbol{2\csc x} \qquad\qquad \boldsymbol{= 2\csc x}$$

Use a half-angle formula to find the exact value of the expression.

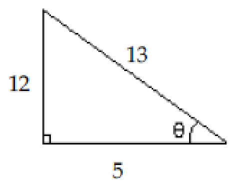
15) tan 105° 15) ____________

$$\tan\left(\frac{\theta}{2}\right) = \frac{1 - \cos\theta}{\sin\theta}$$

$$\tan(105°) = \frac{1 - \cos 210°}{\sin 210°} = \frac{1 - \left(-\frac{\sqrt{3}}{2}\right)}{-\frac{1}{2}} = \frac{\frac{2+\sqrt{3}}{2}}{-\frac{1}{2}} = \mathbf{-2 - \sqrt{3}}$$

Use the figure to find the exact value of the trigonometric function.

16) Find sin 2θ. 16) ____________



$$\sin 2\theta \ = \ 2\sin\theta\cos\theta = 2 \cdot \frac{12}{13} \cdot \frac{5}{13} = \mathbf{\frac{120}{169}}$$

Find all solutions of the equation.

17) cos x = 0 17) ____________

$$\cos x = 0 \text{ at } \left\{x = \cdots,\ -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\right\}$$

We summarize this by saying: $\boldsymbol{x = \frac{\pi}{2} + n\pi}$ Note: n is any integer

Alternatively: $\boldsymbol{x = \frac{\pi}{2} + 2n\pi,\ \frac{3\pi}{2} + 2n\pi}$

Solve the equation on the interval [0, 2π).

18) $\cot^2 x \cos x = \cot^2 x$ 18) ____________

$$\cot^2 x \cos x = \cot^2 x$$

$$\cot^2 x \cos x - \cot^2 x = 0$$

$$\cot^2 x\,(\cos x - 1) = 0$$

$\cot x = 0$ or $\cos x = 1$

$x = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ $x = 0$ (but see below)

$x \neq \{0, \pi\}$ because $\cot x$ does not exist where $\sin x = 0$

Collecting the various solutions, $\boldsymbol{x = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}}$

Note that when the function is set equal to zero, it is not enough to identify where each factor of the function is equal to zero. You must also identify where each factor does not exist and exclude those values from the final solution.

Solve the equation on the interval $[0, 2\pi)$.

19) $\sin 3x = 0$ 19) ____________

This boils down to: $3x = \{0, \pi, 2\pi, 3\pi, 4\pi, 5\pi\}$

Divide this by 3 to get: $\boldsymbol{x = \left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}}$

Note that there are 6 solutions because the usual number of solutions (i.e., 2) is multiplied by the coefficient of the variable, i.e., $k = 3$.

20) $\sin\left(x + \frac{\pi}{6}\right) - \sin\left(x - \frac{\pi}{6}\right) = 1$ 20) ____________

Use the identities:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

$$\sin\left(x + \frac{\pi}{6}\right) - \sin\left(x - \frac{\pi}{6}\right) = 1$$

$$\left(\sin x\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right)\cos x\right) - \left(\sin x\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\cos x\right) = 1$$

$$2\sin\left(\frac{\pi}{6}\right)\cos x = 1$$

$$2 \cdot \frac{1}{2} \cdot \cos x = 1$$

$$\cos x = 1$$

$$\boldsymbol{x = 0}$$

21) $\cos 2x = \frac{\sqrt{2}}{2}$ 21) ____________

This boils down to: $2x = \left\{\frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}\right\}$

Divide this by 2 to get: $\boldsymbol{x = \left\{\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}\right\}}$

Note that there are 4 solutions because the usual number of solutions (i.e., 2) is multiplied by the coefficient of the variable, i.e., $k = 2$.

22) $\cos x + 2\cos x \sin x = 0$ 22) ____________

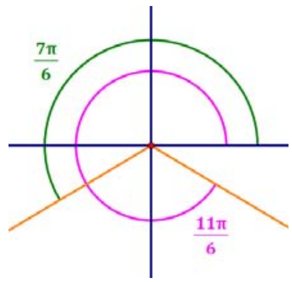
$$\cos x\ (1 + 2\sin x) = 0$$

$\cos x = 0$ or $(1 + 2\sin x) = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\sin x = -\frac{1}{2}$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$



$$\boldsymbol{x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}}$$

Solve the problem.

23) An airplane flying faster than the speed of sound creates sound waves that form a cone. If α is the vertex angle of the cone and m is the Mach number for the speed of the plane, then $\sin\frac{\alpha}{2} = \frac{1}{m}$ $(m > 1)$. Write the formula to calculate the Mach number if $\alpha = 90°$. 23) ____________

$$\sin\left(\frac{\alpha}{2}\right) = \frac{1}{m}$$

$$m = \frac{1}{\sin\left(\frac{\alpha}{2}\right)}$$

$$m = \frac{1}{\sin\left(\frac{90°}{2}\right)}$$

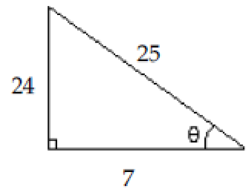
$$m = \frac{1}{\sin(45°)}$$

$$m = \frac{1}{\sqrt{2}/2}$$

$$\boldsymbol{m = \sqrt{2}}$$

Use the figure to find the exact value of the trigonometric function.

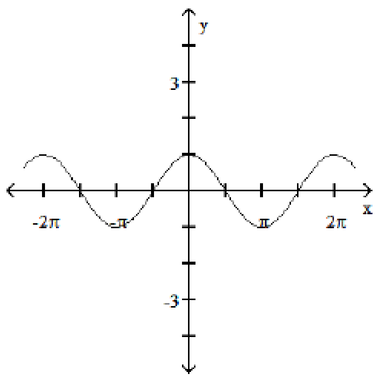
24) Find tan 2θ. 24) ____________



$$\tan\ 2\theta\ = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\cdot\left(\frac{24}{7}\right)}{1-\left(\frac{24}{7}\right)^2} = \frac{\frac{48}{7}}{\frac{-527}{49}} = -\frac{\mathbf{336}}{\mathbf{527}}$$

Use the graph to complete the identity.

25) $\frac{(\sec x + \tan x)(\sec x - \tan x)}{\sec x} = ?$ 25) ____________



This function looks like $y = \cos x$, so let's try to show:

$$\frac{(\sec x + \tan x)(\sec x - \tan x)}{\sec x} \qquad = \mathbf{\cos x}$$

$$\frac{\sec^2 x - \tan^2 x}{\sec x}$$

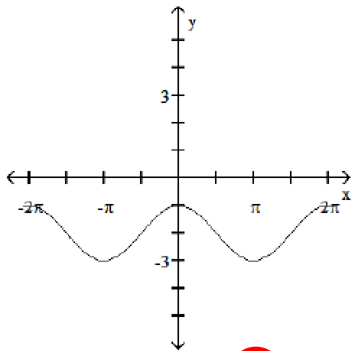
$$\frac{(1+\tan^2 x) - \tan^2 x}{\sec x}$$

$$\frac{1}{\sec x}$$

$$\mathbf{\cos x} \qquad = \mathbf{\cos x}$$

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

26) $\dfrac{\cos x \tan x - 2\tan x + 5\cos x - 10}{\tan x + 5} = ?$ 26) ______



A) $\sin x - 5$ B) $\cos x - 2$ C) $\sin x + 5\cos x$ D) $\cos x + 2$

$$\frac{\cos x \tan x - 2\tan x + 5\cos x - 10}{\tan x + 5} = \frac{\tan x\,(\cos x - 2) + 5(\cos x - 2)}{\tan x + 5}$$

$$= \frac{(\tan x + 5)\,(\cos x - 2)}{\tan x + 5}$$

$$= \cos x - 2$$

Answer B

Note: The graph of this function should have holes in all locations where $\tan x = -5$, since at these points the denominator of the expression would be zero. I don't see any holes in the graph, but maybe they are too small for me to see. ☺