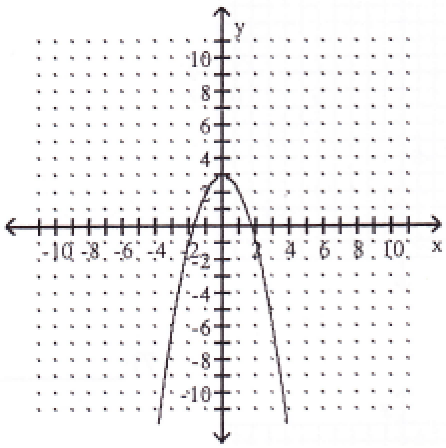
A note on terminology: Zeros and roots are the same thing. If they are real, as opposed to complex, they are also x-intercepts of your graph.

The graph of a quadratic function is given. Determine the function's equation.

1)



A) $f(x) = -x^2 - 6x - 9$ B) $g(x) = -x^2 + 6x + 9$

C) $j(x) = -x^2 + 3$ D) $h(x) = -x^2 - 3$

Use the roots of the equation and the direction that the curve opens determine the equation. It looks like the roots are around ± 2, but a quick look at the answers tells us that there are no 2's. So the numbers around ± 2 must be $\pm\sqrt{3}$ (i.e. $\sim \pm 1.732$). Let's try that:

$$(x+\sqrt{3})(x-\sqrt{3}) = x^2 - 3$$

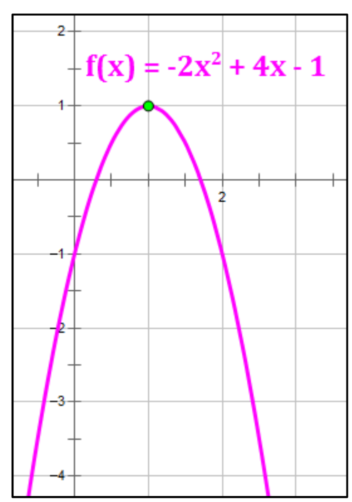
Next, note that the curve is an inverted-U, so that the lead coefficient must be negative. Then,

$$-(x^2-3) = -x^2+3 \qquad \textbf{Answer C}$$

Find the coordinates of the vertex for the parabola defined by the given quadratic function.

2) $f(x) = -2x^2 + 4x - 1$

For a quadratic equation in the form $ax^2 + bx + c = 0$, the x-coordinate of the vertex occurs at $x = -\frac{b}{2a}$. How can you remember this? Recall that the x-coordinate of the vertex occurs at the average of the two roots of the quadratic function.



$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

There it is: $x = -\frac{b}{2a}$.

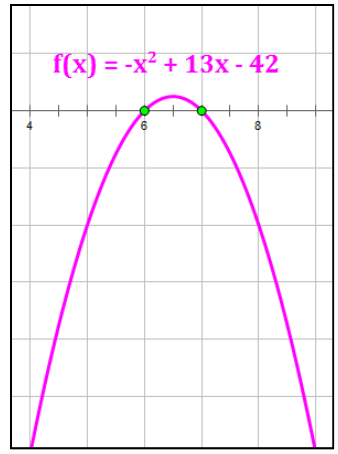
For this problem, we have: $a = -2 \quad b = 4 \quad c = -1$

$$x = -\frac{4}{2(-2)} = 1 \qquad y = f(1) = -2(1)^2 + 4(1) - 1 = 1$$

So, the vertex is at $(\mathbf{1}, \mathbf{1})$

Find the x-intercepts (if any) for the graph of the quadratic function. Give your answers in exact form.

3) $f(x) = -x^2 + 13x - 42$



It looks like this function can be factored, which is the fastest way to find x-intercepts, which are also called roots or zeros.

Starting Equation: $-x^2 + 13x - 42 = 0$

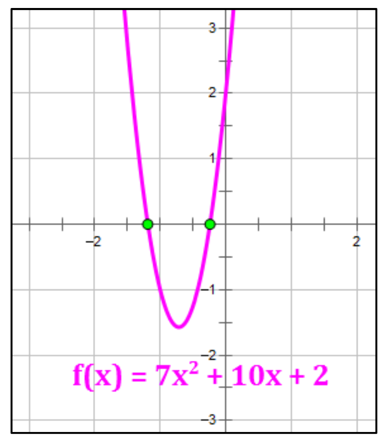
Factor out (-1): $-(x^2 - 13x + 42) = 0$

Factor the remaining trinomial: $-(x-6)(x-7) = 0$

Break into separate equations: $x - 6 = 0 \quad x - 7 = 0$

Identify x-intercepts: $\boldsymbol{x = \{6, 7\}}$

4) $7x^2 + 10x + 2 = 0$



This one is ugly. Let's use the quadratic formula:

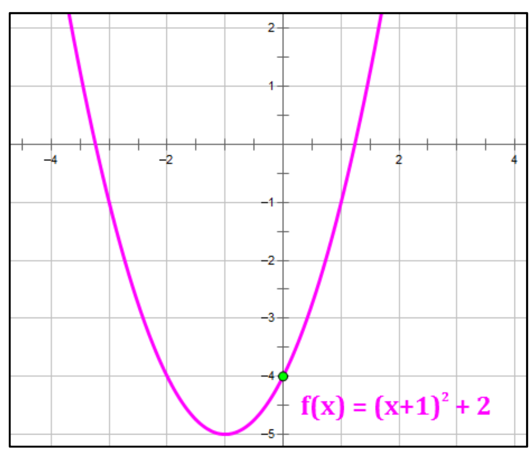
Note that: $a = 7$, $b = 10$, $c = 2$. Then,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-10 \pm \sqrt{10^2 - 4(7)(2)}}{2(7)}$$

$$= \frac{-10 \pm \sqrt{44}}{14} = \frac{-10 \pm 2\sqrt{11}}{14} = \boldsymbol{\frac{-5 \pm \sqrt{11}}{7}}$$

Find the y-intercept for the graph of the quadratic function.

5) $f(x) = (x + 1)^2 - 5$



The y-intercept is $f(0)$.

$$f(x) = (x+1)^2 - 5$$

$$f(0) = (0+1)^2 - 5$$

$$= 1 - 5 = \boldsymbol{-4}$$

Use the vertex and intercepts to sketch the graph of the quadratic function.

6) $y - 1 = (x + 3)^2$

This equation is in vertex form: $y - k = a(x - h)^2$, where (h, k) is the vertex.

Therefore, the vertex of this function is $(-3, 1)$.

To get the intercepts, you can expand the function.

Starting Equation:	$y - 1 = (x + 3)^2$
Add 1:	$y = (x + 3)^2 + 1$
Expand the square term:	$y = x^2 + 6x + 9 + 1 = x^2 + 6x + 10$

Use the quadratic formula: $a = 1$, $b = 6$, $c = 10$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{6^2 - 4(1)(10)}}{2(1)} = \frac{-6 \pm \sqrt{-4}}{2} = \frac{-6 \pm 2i}{2} = \mathbf{-3 \pm i}$$

Conclusion: There are no real x-intercepts. While it is possible to use a function's complex roots to develop a graph, the method to do this is not usually taught in high school math. To see how this is done, check out the Algebra App on www.mathguy.us. Let's use a couple of points to help us finish the graph.

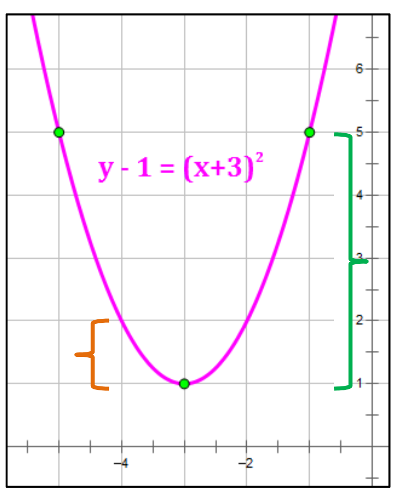
$f(x) = (x + 3)^2 + 1$

$f(-1) = (-1 + 3)^2 + 1 = 4 + 1 = \mathbf{5}$ So a point is $(-1, 5)$.

$f(-5) = (-5 + 3)^2 + 1 = 4 + 1 = \mathbf{5}$ So a point is $(-5, 5)$.

Plot the vertex and these two points and run a smooth curve through them. Notice a couple of things:

Moving one unit left or right from the vertex yields a point $a\ (= 1)$ unit(s) away from the vertex in the vertical direction.



Moving two units left or right from the vertex yields a point $4a\ (= 4)$ units away from the vertex in the vertical direction.

7) $f(x) = x^2 + 6x + 5$

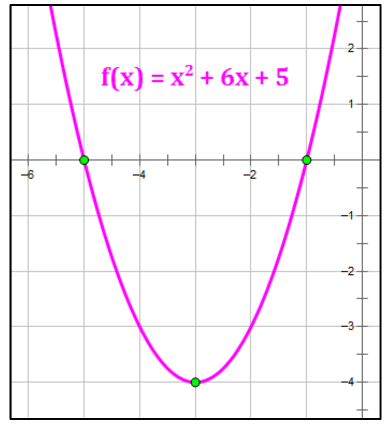
This equation is easy enough to factor and find the x-intercepts.

Starting Equation: $y = x^2 + 6x + 5$

Factor the trinomial $y = (x+5)(x+1)$

Break into separate equations: $x + 5 = 0 \quad x + 1 = 0$

Identify x-intercepts: $x = \{-5, -1\}$



The vertex occurs at $x = -\frac{b}{2a}$.

For this problem, we have: $a = 1 \quad b = 6 \quad c = 5$

$$x = -\frac{6}{2(1)} = -3$$

$$y = f(-3) = (-3)^2 + 6(-3) + 5 = -4$$

So, the vertex is at $(-3, -4)$. Plot the vertex and the two x-intercepts, and run a smooth curve through them.

8) $f(x) = 3x^2 - 12x + 15$

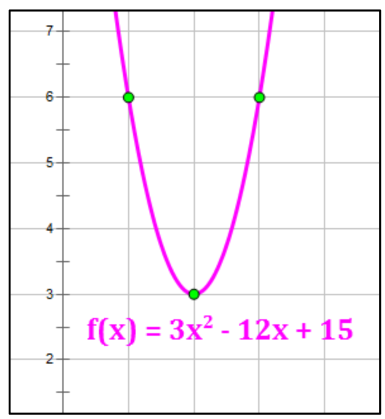
The vertex occurs at $x = -\frac{b}{2a}$.

For this problem, we have: $a = 3 \quad b = -12 \quad c = 15$

$$x = -\frac{-12}{2(3)} = 2$$

$$y = f(2) = 3(2)^2 - 12(2) + 15 = 3$$

So, the vertex is at $(2, 3)$



Instead of using the full quadratic formula this time, let's check whether there are any real x-intercepts using the discriminant:

$$a = 3, \quad b = -12, \quad c = 15$$

$$\Delta = b^2 - 4ac = (-12)^2 - 4(3)(15) = -36$$

A negative discriminant implies that there are no real roots and, therefore, no real x-intercepts. So, let's use a couple of points to help us finish the graph.

$$f(x) = 3x^2 - 12x + 15$$

$f(1) = 3(1)^2 - 12(1) + 15 = \mathbf{6}$ So a point is $(1, 6)$.

$f(3) = 3(3)^2 - 12(3) + 15 = \mathbf{6}$ So a point is $(3, 6)$.

Plot the vertex and these two points, and run a smooth curve through them.

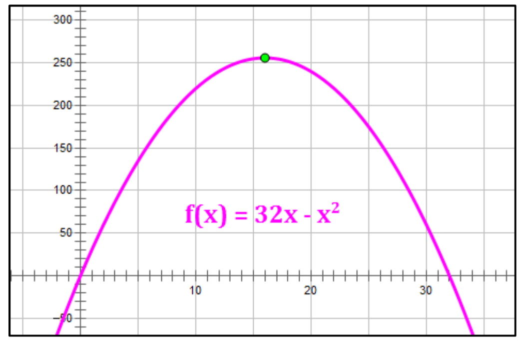
9) Among all pairs of numbers whose sum is 32, find a pair whose product is as large as possible.

Mathematically, we can find the answer in the following manner:

Let the two numbers be x and $32 - x$.

Their product forms a parabola, of which we want to find the maximum. So,

$$f(x) = x(32 - x) = 32x - x^2$$



We find the maximum value of $f(x)$ at the vertex.

$$a = -1 \qquad b = 32 \qquad c = 0$$

$$x = -\frac{b}{2a} = -\frac{32}{2(-1)} = 16$$

So, the solution is the pair of numbers 16 and $(32 - 16) = 16$. i.e., **16 and 16**

Note this is a common problem that takes a number of forms in high school mathematics. Unless there is a twist to the problem, the maximum is always obtained by two numbers that are each half of the original sum. In this case, $32 \div 2 = 16$, so both numbers are 16.

Find the zeros for the polynomial function and give the multiplicity for each zero.

10) $f(x) = 2(x + 4)(x - 3)^2$

Easy peezy. Look at the factors to find the zeros (roots):

$(x + 4)$ occurs with an exponent of one, so the zero $x = -4$ has multiplicity 1.

$(x - 3)$ occurs with an exponent of two, so the zero $x = 3$ has multiplicity 2.

11) $f(x) = x^3 + x^2 - 30x$

Let's factor it, then the answer is easy peezy.

$$x^3 + x^2 - 30x = x(x^2 + x - 30) = x(x + 6)(x - 5)$$

(x) occurs with an exponent of one, so the zero $x = 0$ has multiplicity 1.

$(x + 6)$ occurs with an exponent of one, so the zero $x = -6$ has multiplicity 1.

$(x - 5)$ occurs with an exponent of one, so the zero $x = 5$ has multiplicity 1.

12) $f(x) = x^3 + 8x^2 - x - 8$

Let's factor it, then the answer is easy peezy.

$x^3 + 8x^2 - x - 8 = (x^3 + 8x^2) - (x + 8)$ Be careful with your signs on this step!

$= x^2(x + 8) - 1(x + 8)$

$= (x^2 - 1)(x + 8) = (x - 1)(x + 1)(x + 8)$

$(x - 1)$ occurs with an exponent of one, so the zero $x = 1$ has multiplicity 1.

$(x + 1)$ occurs with an exponent of one, so the zero $x = -1$ has multiplicity 1.

$(x + 8)$ occurs with an exponent of one, so the zero $x = -8$ has multiplicity 1.

Graph the polynomial function.

13) $f(x) = x^3 + 5x^2 - x - 5$

Let's factor it to find the roots.

$x^3 + 5x^2 - x - 5 = (x^3 + 5x) - (x + 5)$ Be careful with your signs on this step!

$= x^2(x + 5) - 1(x + 5)$

$= (x^2 - 1)(x + 5) = (x - 1)(x + 1)(x + 5)$

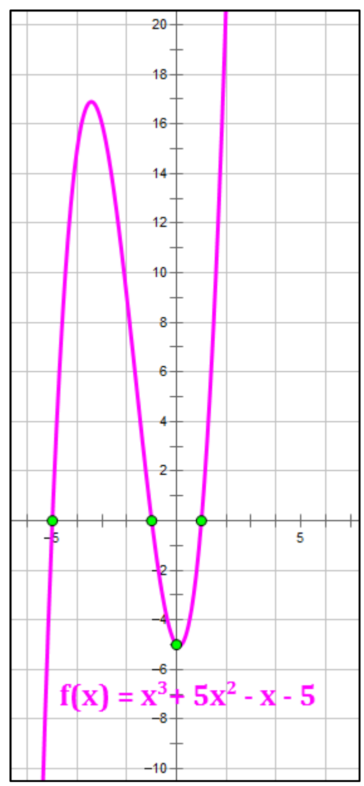
The x-intercepts then are: $x = \{1, -1, -5\}$

We also know that for a cubic function with a positive lead coefficient:

- As $x \to \infty$, $f(x) \to \infty$.
- As $x \to -\infty$, $f(x) \to -\infty$.

Finally, we know the graph has a y-intercept of $f(0) = -5$.

So, our graph starts at $-\infty$, crosses the x-axis three times, at $x = \{1, -1, -5\}$, and then shoots off to $+\infty$. It also contains the point $(0, -5)$. It looks something like the graph at right.



Shameless self-promotion: the Algebra App, available at www.mathguy.us will graph any polynomial with 13 or fewer terms, identify the zeros, and much more. Check it out.

Perform the Indicated Operation.

Divide using long division.

14) $(6x^3 - 2) \div (3x - 1)$

$$2x^2 + \frac{2}{3}x + \frac{2}{9} - \frac{16}{9(3x+1)}$$

$$\begin{array}{r|l} 3x - 1 & 6x^3 + 0x^2 + 0x - 2 \\ & \underline{6x^3 - 2x^2} \\ & \quad 2x^2 + 0x - 2 \\ & \quad \underline{2x^2 - \frac{2}{3}x} \\ & \qquad \frac{2}{3}x - 2 \\ & \qquad \underline{\frac{2}{3}x - \frac{2}{9}} \\ & \qquad\quad -\frac{16}{9} \end{array}$$

Yes, we hate long division of a polynomial, favoring the much more concise and probably more accurate (for most of us) synthetic division. However, we must follow the instructions.

15) $(3x^5 - x^3 - 2x^2 - 7x + 4) \div (x^2 - 2)$

$$3x^3 + 5x - 2 + \frac{3x}{x^2 - 2}$$

$$\begin{array}{r|l} x^2 - 2 & 3x^5 + 0x^4 - 1x^3 - 2x^2 - 7x + 4 \\ & \underline{3x^5 \quad - 6x^3} \\ & \qquad 5x^3 - 2x^2 - 7x + 4 \\ & \qquad \underline{5x^3 \quad - 10x} \\ & \qquad\quad -2x^2 + 3x + 4 \\ & \qquad\quad \underline{-2x^2 \quad + 4} \\ & \qquad\qquad 3x \end{array}$$

16) $\dfrac{x^4 + 3x^3 + x^2 + 6x + 3}{x + 1}$

$$\begin{array}{r|l} & x^3 + 2x^2 - x + 7 - \dfrac{4}{x+1} \\ \hline x+1 & x^4 + 3x^3 + 1x^2 + 6x + 3 \\ & \underline{x^4 + x^3} \\ & 2x^3 + 1x^2 + 6x + 3 \\ & \underline{2x^3 + 2x^2} \\ & -1x^2 + 6x + 3 \\ & \underline{-1x^2 - 1x} \\ & 7x + 3 \\ & \underline{7x + 7} \\ & -4 \end{array}$$

Divide using synthetic division.

17) $(x^5 - 4x^4 - 9x^3 + x^2 - x + 21) \div (x + 2)$

In synthetic division, use the root of the divisor as your multiplier.

$$\begin{array}{r|rrrrrr} & x^5 & x^4 & x^3 & x^2 & x & c \\ -2 & 1 & -4 & -9 & 1 & -1 & 21 \\ & & -2 & 12 & -6 & 10 & -18 \\ \hline & 1 & -6 & 3 & -5 & 9 & 3 \\ & x^4 & x^3 & x^2 & x & c & rem \end{array}$$

Result:

$$\boldsymbol{x^4 - 6x^3 + 3x^2 - 5x + 9 + \frac{3}{x+2}}$$

Use synthetic division to show that the number given to the right of the equation is a solution of the equation, then solve the polynomial equation.

18) $x^3 - 2x^2 - 5x + 6 = 0;\ 3$

In synthetic division, use the root as your multiplier.

$$\begin{array}{r|rrrr} & x^3 & x^2 & x & c \\ 3 & 1 & -2 & -5 & 6 \\ & & 3 & 3 & -6 \\ \hline & 1 & 1 & -2 & \mathbf{0} \\ & x^2 & x & c & rem \end{array}$$

When the remainder is zero, the multiplier used (3) is a solution of the equation (and is a zero, or root, of the polynomial).

Result: $x^2 + x - 2 = (x+2)(x-1)$ so, -2 and 1 are zeros, as is 3 from above.

$$\boldsymbol{x = \{-2, 1, 3\}}$$

Find a rational zero of the polynomial function and use it to find all the zeros of the function.

19) $f(x) = 3x^3 - 17x^2 + 18x + 8$

We need to find the zeros of this polynomial. What can we use to be smart about this?

- Since its degree is odd, there must be at least one real zero.
- Any rational zeros must take the form p/q where p is a factor of the constant term (8), and q is a factor of the lead coefficient (3). So possible rational zeros are:

  $$\pm\frac{8}{3}\ \ \pm\frac{4}{3}\ \ \pm\frac{2}{3}\ \ \pm\frac{1}{3}\ \ \pm 8\ \ \pm 4\ \ \pm 2\ \ \pm 1.$$

- If 1 is a zero, the sum of the coefficients is zero; this is not the case for this problem.
- Descartes' Rule of Signs is not much help since there are multiple sign changes.

Next step: one or more synthetic divisions. You could try either 2 or 4 (you don't need both).

$$\begin{array}{r|rrrr} & x^3 & x^2 & x & c \\ 2 & 3 & -17 & 18 & 8 \\ & & 6 & -22 & -8 \\ \hline & 3 & -11 & -4 & 0 \\ & x^2 & x & c & rem \end{array}$$ bingo!

OR

$$\begin{array}{r|rrrr} & x^3 & x^2 & x & c \\ 4 & 3 & -17 & 18 & 8 \\ & & 12 & -20 & -8 \\ \hline & 3 & -5 & -2 & 0 \\ & x^2 & x & c & rem \end{array}$$ bingo!

Result: $3x^2 - 5x - 2 = (3x+1)(x-2)$ so, $-\frac{1}{3}$ and 2 are zeros, as is 4 from above.

$$\boldsymbol{x = \left\{-\frac{1}{3}, 2, 4\right\}}$$

20) $f(x) = x^3 + 5x^2 + 17x + 13$

We need to find the zeros of this polynomial. What can we use to be smart about this?

- Since its degree is odd, there must be at least one real zero.
- Any rational zeros must take the form p/q where p is a factor of the constant term (13), and q is a factor of the lead coefficient (1). So possible rational zeros are:

  $\pm 1, \pm 13.$
- If 1 is a zero, the sum of the coefficients is zero; this is not the case for this problem.
- If -1 is a zero, sums of alternating coefficients are equal. This is the case here, since $1 + 17 = 5 + 13$. Therefore, -1 is a zero of this polynomial.
- Descartes' Rule of Signs tells us there are no positive roots (there are no sign changes in the polynomial).

Next step: synthetic division by -1.

$$\begin{array}{r|rrrrl}
 & x^3 & x^2 & x & c & \\
-1 & 1 & 5 & 17 & 13 & \\
 & & -1 & -4 & -13 & \\
\hline
 & 1 & 4 & 13 & 0 & \text{bingo!} \\
 & x^2 & x & c & rem &
\end{array}$$

Result: $x^2 + 4x + 13$ Yuk! Let's use the quadratic formula:

Note that: $a = 1$, $b = 4$, $c = 13$. Then,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{4^2 - 4(1)(13)}}{2(1)}$$

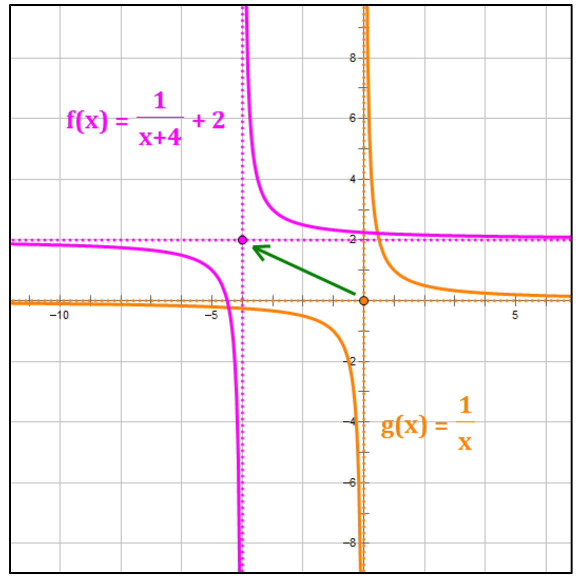
$$= \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6\mathrm{i}}{2} = \boldsymbol{-2 \pm 3i}$$

So, $-2 \pm 3i$ are zeros, as is -1 from above.

$$\boldsymbol{x = \{-1, -2 \pm 3i\}}$$

Use transformations of $f(x) = \frac{1}{x}$ or $f(x) = \frac{1}{x^2}$ to graph the rati

21) $f(x) = \frac{1}{x + 4} + 2$



We are asked to graph this function using transformation. That means we need to start with the parent function and translate the function to its new position.

The parent function, shown in the illustration at right in orange is: $f(x) = \frac{1}{x}$.

The general form of a simple rational function is: $f(x) = \frac{1}{x-h} + k$, where (h, k) is the point at the intersection of its asymptotes. The general form also indicates the nature of the translation, which is h units to the left and k units up.

For the function given in this problem, we translate the curve 4 units left $(h = -4)$ and 2 units up $(k = 2)$. This can be accomplished by translating the asymptotes for the parent function, as shown in the illustration by a green arrow. The resulting graph is shown in magenta. The intersection of its asymptotes occurs at the point $(h, k) = (-4, 2)$.

Note: An asymptote is a line that a function approaches, but never reaches. For a simple rational function of the form $f(x) = \frac{1}{x-h} + k$, the asymptotes occur at $x = h$ and $y = k$.

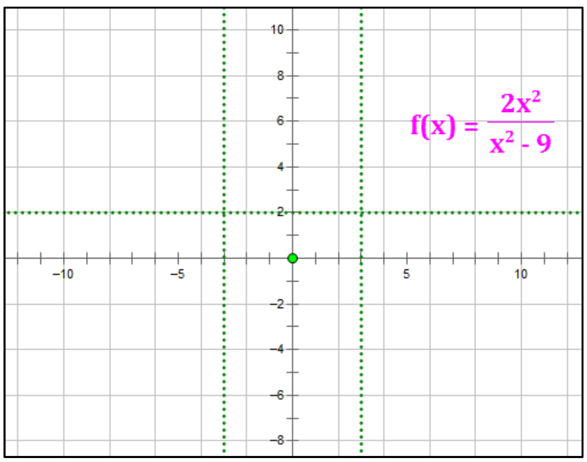
Graph the rational function.

22) $f(x) = \dfrac{2x^2}{x^2 - 9}$

This function has:

- Vertical asymptotes where the denominator, $x^2 - 9 = 0$, i.e., at $x = \pm 3$.
- A horizontal asymptote at $y = \lim_{x\to\infty} \frac{2x^2}{x^2-9} = \frac{2x^2}{x^2} = 2$
- A point at $(0, 0)$, since $f(0) = 0$.

First draw the guides (asymptotes and the identified point).



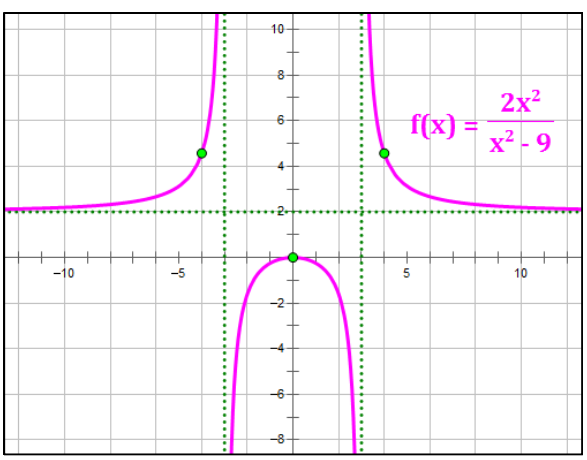
So far, so good, but it looks like we would benefit from finding a point on the left and right sides of the vertical asymptotes.

$$f(4) = \frac{2(4)^2}{(4)^2 - 9} = \frac{32}{7} = 4\frac{4}{7}$$

$$f(-4) = \frac{2(-4)^2}{(-4)^2 - 9} = \frac{32}{7} = 4\frac{4}{7}$$

So two points are: $(4, 4\frac{4}{7})$ and $(-4, 4\frac{4}{7})$.

Plot these points and then draw in the function.



Note: if you need more points to visualize what the function looks like, select some x-values and calculate the corresponding y-values wherever you think you need them.

Graph the function.

23) $f(x) = \dfrac{x^2 + 4x - 9}{x - 8}$

This function has a vertical asymptote where the denominator, $x - 8 = 0$, i.e., at $x = 8$.

It also has a slant asymptote at: $y = \frac{x^2+4x-9}{x-8}$, excluding the remainder, so let's divide and find the slant asymptote.

$$\begin{array}{c|ccc} & x^2 & x & c \\ 8 & 1 & 4 & -9 \\ & & 8 & 96 \\ \hline & 1 & 12 & -87 \\ & x & c & rem \end{array}$$

The remainder (-87) is ignored in developing the equation of the slant asymptote, which is $y = x + 12$.

Finally, let's calculate a couple of points to help nail down the curve.

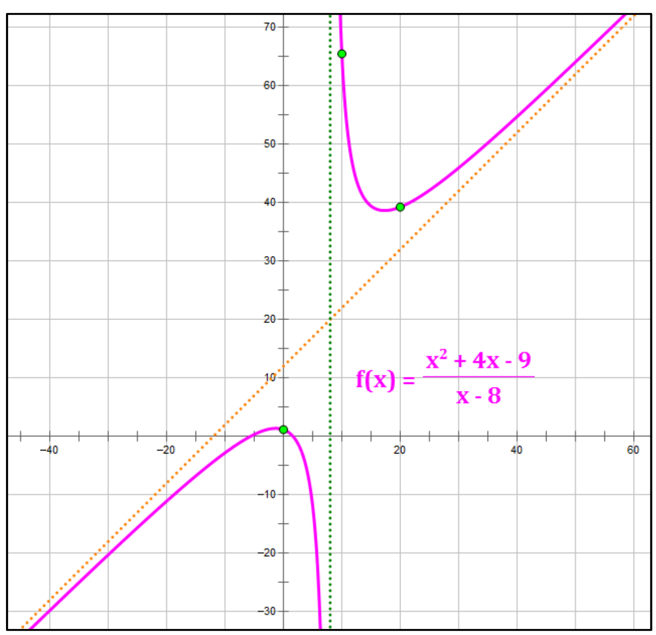
$$f(0) = \frac{0^2 + 4(0) - 9}{0 - 8} = \frac{-9}{-8} = 1\frac{1}{8} \quad \Rightarrow \quad (0, 1\tfrac{1}{8})$$

$$f(10) = \frac{(10)^2 + 4(10) - 9}{10 - 8} = \frac{131}{2} = 65\frac{1}{2} \quad \Rightarrow \quad (10, 65\tfrac{1}{2})$$

$$f(20) = \frac{(20)^2 + 4(20) - 9}{20 - 8} = \frac{471}{12} = 39\frac{1}{4} \quad \Rightarrow \quad (20, 39\tfrac{1}{4})$$

You may not need to calculate all three points to generate a curve acceptable to your teacher. I did it here so you could see how it works.

Then, plot the asymptotes, the points, and finally, the curve.



Find the slant asymptote, if any, of the graph of the rational function.

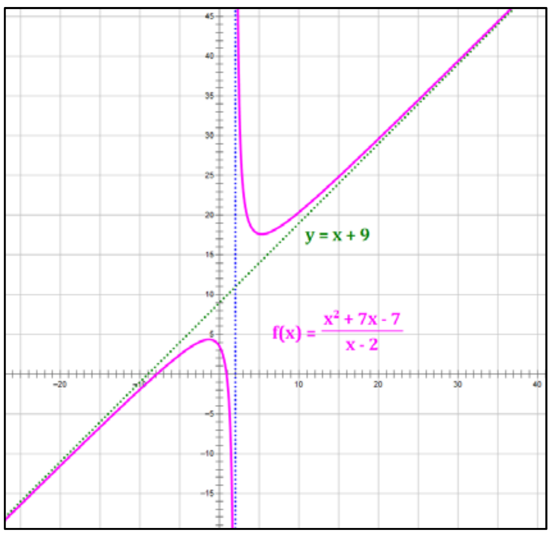
24) $f(x) = \dfrac{x^2 + 7x - 7}{x - 2}$

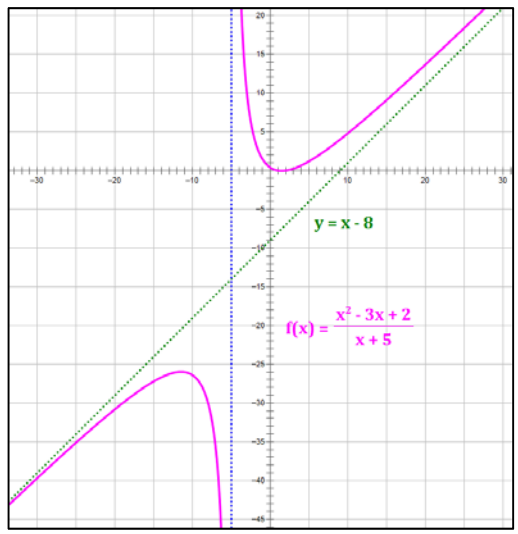
The slant asymptote is found by dividing the rational function and discarding the remainder.

$$y = \frac{x^2 + 7x - 7}{x - 2}$$

$$\begin{array}{r|rrr} & x^2 & x & c \\ 2 & 1 & 7 & -7 \\ & & 2 & 18 \\ \hline & 1 & 9 & 11 \\ & x & c & rem \end{array}$$

The remainder (11) is ignored in developing the equation of the slant asymptote, which is: $\boldsymbol{y = x + 9}$.





25) $f(x) = \dfrac{x^2 - 3x + 2}{x + 5}$

The slant asymptote is found by dividing the rational function and discarding the remainder.

$$y = \frac{x^2 - 3x + 2}{x + 5}$$

$$\begin{array}{r|rrr} & x^2 & x & c \\ -5 & 1 & -3 & 2 \\ & & -5 & 40 \\ \hline & 1 & -8 & 42 \\ & x & c & rem \end{array}$$

The remainder (42) is ignored in developing the equation of the slant asymptote, which is: $\boldsymbol{y = x - 8}$.

A Learning Extension for the Curious. Much more than the slant asymptote.

The slant asymptote is the simplest in a class of asymptotes that are not vertical or horizontal. Rational functions may have 2nd, 3rd, 4th and higher degree curves as asymptotes. In general, the degree of the asymptote is equal to the degree of the numerator minus the degree of the denominator. Consider the following rational function, inspired by Problem 15 above:

$$f(x) = \frac{3x^5 - x^3 - 2x^2 - 7x + 4}{x^2 - 2}$$

This curve has vertical asymptotes at $x = \sqrt{2}$ and $x = -\sqrt{2}$ because of the denominator.

To obtain the asymptotic curve, divide the rational function and discard the remainder.

$$f(x) = \frac{3x^5 - x^3 - 2x^2 - 7x + 4}{x^2 - 2} = 3x^3 + 5x - 2 + \frac{3x}{x^2 - 2} \quad \text{(from Problem 15)}$$

The result of the division provides the asymptotic curve, which in this case is the cubic function:

$$g(x) = 3x^3 + 5x - 2$$

Let's look at this graphically. Notice how the magenta curve closes in on the green curve on the left and right sides of the graph. Very cool stuff! Now, you are smarter!