Quaternions

Quaternions were invented by William Rowan Hamilton on 1843 as an extension of the set of Complex numbers. The set of Quaternions is considered to be a Division Algebra, and has its own set of rules.

Complex number: $a + bi$, where $i^2 = -1$.

Quaternion: $a + bi + cj + dk$, where $i^2 = j^2 = k^2 = ijk = -1$.

The ideas underlying quaternions were percolating in Hamilton's mind on the way to a meeting with the Irish Royal Academy in Dublin. As he walked along the bank of the Royal Canal with his wife, he has his Eureka moment on Quaternions, and carved the formula above in the stone of the Brougham Bridge.

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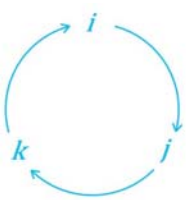
The set of Quaternions is denoted by some variant of the letter $\boldsymbol{H}$. The set of rules Quaternions follow are:

(1) $i^2 = j^2 = k^2 = ijk = -1$.

(2) $ij = k, \; jk = i, \; ki = j$

(3) $ji = -k, \; kj = -i, \; ik = -j$



Quaternion multiplication of i, j, and k can be described by the diagram to the left. Multiplication of two of these gives the third as a positive result when in the direction of the arrows (clockwise) and a negative result when in the opposite direction (counter-clockwise). Note that the quaternions do not have a commutative property of multiplication.

Exercises: Show that:

- $i \cdot (jk) = (ij) \cdot k$
- $(ij)^2 \neq i^2 \cdot j^2$
- The set of Quaternions is closed under multiplication. That is, show that the product of any two quaternions of the form $a + bi + cj + dk$ results in another quaternion of that form.

The following 2x2 Complex matrices form a set of Quaternions:

$$\mathrm{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \mathrm{I} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \; \mathrm{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; \mathrm{K} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Show that:

- $\mathrm{I}^2 = \mathrm{J}^2 = \mathrm{K}^2 = -\mathrm{U}$
- $\mathrm{I} \cdot \mathrm{J} = \mathrm{K}$

What are Quaternions used for?

Quaternions can be considered a of ordered quadruples in four-dimensional Real or Complex space. The constant component of a quaternion can be a magnitude or force, and the i, j, and k components can be an ordered triple in space.

The next time you are admiring the smooth animations in computer games or movies, you may have Quaternions to thank. They have been used to solve a number of problems in animation, such as gimbal lock (alignment of two axes of yaw, roll and pitch) and instability caused by interpolation.

A Google search of "quaternions applications" turns up a number of articles on spatial rotations. In an April 28, 2015 paper, The Quaternions and their Applications, Rob Eimerl identified the following real-life applications for Quaternions:

- Group Theory: The quaternions form an order 8 subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$. [This is why the Quaternions are sometimes referred to as Q_8.]
- Number Theory: The mathematician Hurwitz used Quaternions to prove Lagrange's theorem, that every positive integer is a sum of at most four squares.
- Rotations: Quaternions can describe rotations in 3-dimensional space.
- Computer Graphics: Quaternions generate a more realistic animation. A technique which is currently gaining favor is called spherical linear interpolation (SLERP) and uses the fact that the set of all unit quaternions form a unit sphere. By representing the quaternions of key frames as points on the unit sphere, a SLERP defines the intermediate sequence of rotations as a path along the great circle between the two points on the sphere.
- Physics: Quaternions have found use in a wide variety of research.
  - They can be used to express the Lorentz Transform making them useful for work on Special and General Relativity.
  - Their properties as generators of rotation make them useful for Newtonian Mechanics, scattering experiments such as crystallography, and quantum mechanics.