Solving Simultaneous Equations

One method of solving simultaneous equations is through the use of an Augmented Matrix. A matrix is considered augmented if it consists of the matrix of the coefficients of the variables, augmented by the constant terms. In order for a system of equations to be solved in this form, they must be written in standard form.

Example:

To solve the system:

$$\begin{aligned} 2x + 3y - 2z &= -5 \\ x - y + 2z &= 9 \\ -x + 2y + 4z &= 8 \end{aligned}$$

The augmented matrix would be:

$$\left[\begin{array}{ccc|c} 2 & 3 & -2 & -5 \\ 1 & -1 & 2 & 9 \\ -1 & 2 & 4 & 8 \end{array}\right]$$

Gauss-Jordan Elimination

A process called Gauss-Jordan Elimination (GJE) is used to manipulate the augmented matrix to obtain a solution to the equations. GJE is also called Row Reduction because each step adjusts the values in one row of the augmented matrix. At the end of the process, the rows of the coefficient matrix are “reduced” to the Identity Matrix.

The following manipulations of the rows are allowed:

- Multiplying or dividing a row by a scalar (i.e., a number).
- Switching any two rows.
- Adding or subtracting a multiple of one row to or from another.

When this process is complete, the constant column of the augmented matrix has been converted to the solution of the system of equations. Why does this work? The process used is essentially the same as solving a system of equations by the elimination method. In GJE, you ignore the variable names by using matrices, but the manipulations are the same.

Developing an Inverse Matrix

This process can also be used to develop an Inverse Matrix. To do this,

- Place an identity matrix to the right of the augmented matrix at the start.
- Perform all row operations on this matrix as you progress.
- At the end, the original identity matrix will have been converted to the inverse matrix.

Example:

Solve:

$$\begin{aligned} 2x + 3y - 2z &= -5 \\ x - y + 2z &= 9 \\ -x + 2y + 4z &= 8 \end{aligned}$$

1. **Initial Augmented Matrix**

$$\left[\begin{array}{ccc|c} 2 & 3 & -2 & -5 \\ 1 & -1 & 2 & 9 \\ -1 & 2 & 4 & 8 \end{array}\right]$$

2. **Switch Rows 1 and 2**

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 2 & 3 & -2 & -5 \\ -1 & 2 & 4 & 8 \end{array}\right]$$

3. **Subtract (2 * Row 1) from Row 2**
   Add Row 1 to Row 3

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 5 & -6 & -23 \\ 0 & 1 & 6 & 17 \end{array}\right]$$

4. **Switch Rows 2 and 3**

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & 6 & 17 \\ 0 & 5 & -6 & -23 \end{array}\right]$$

5. **Subtract (5 * Row 2) from Row 3**

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & 6 & 17 \\ 0 & 0 & -36 & -108 \end{array}\right]$$

6. **Divide Row 3 by -36**

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & 6 & 17 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

7. **Subtract (6 * Row 3) from Row 2**
   Subtract (2 * Row 3) from Row 1

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

8. **Add Row 2 to Row 1**

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Solutions

$x = 2$

$y = -1$

$z = 3$

Notes:

1. Gaussian Elimination is completed after Step 6.
2. The matrix in Step 6 is in Row Echelon Form.
3. Gauss-Jordan Elimination is completed after Step 8.
4. The matrix in Step 6 is in Reduced Row Echelon Form.

You Try It (Solution):

Solve:

$$\begin{aligned} -x - y + z &= -1 \\ 2x + y - 2z &= -2 \\ x + 2y - 3z &= 1 \end{aligned}$$

1. **Initial Augmented Matrix**

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & -1 \\ 2 & 1 & -2 & -2 \\ 1 & 2 & -3 & 1 \end{array}\right]$$

2. **Multiply Row 1 by -1**

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 1 & -2 & -2 \\ 1 & 2 & -3 & 1 \end{array}\right]$$

3. **Subtract (2 * Row 1) from Row 2**
   Subtract Row 1 from Row 3

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -1 & 0 & -4 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

4. **Multiply Row 2 by -1**

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

5. **Subtract Row 2 from Row 3**

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

6. **Divide Row 3 by -2**

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

7. **Add Row 3 to Row 1**

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

8. **Subtract Row 2 from Row 1**

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solutions

$x = -1$
$y = 4$
$z = 2$

Who were Gauss and Jordan?

Johann Carl Friedrich Gauss (30 April 1777 Braunschweig – 23 February 1855 Göttingen) was a German mathematician and astronomer who is ranked as one of history's most influential mathematicians. Often referred to as the "Prince of Mathematicians" and "greatest mathematician since antiquity", he made significant contributions to several fields including number theory, algebra, statistics, analysis, geometry, astronomy, and matrix theory.

Born to poor working-class parents in Brunswick, he started displaying evidence of his genius while he was just a young child. A child prodigy, he is said to have corrected an error in his father's payroll calculations as a small boy of three. He began to astonish his teachers with his brilliance at school and made his first ground-breaking mathematical discovery while he was still a teenager. Even though his parents were poor, he found a patron in the Duke of Brunswick who recognized his intelligence and sent him to the prestigious University of Göttingen.

Eventually he established himself as a prominent mathematician in Germany and his reputation soon spread internationally. He made notable contributions to almost all fields in mathematics, but his favorite area was number theory, a field which he revolutionized with his work on complex numbers. He also published many books including 'Disquisitiones Arithmeticae' which is regarded as one of the most influential mathematics books ever written.

Source: www.thefamouspeople.com/profiles/carl-f-gauss-442.php

Wilhelm Jordan (1 March 1842, Ellwangen, Württemberg – 17 April 1899, Hanover) was a German geodesist (the branch of mathematics dealing with the shape and area of the earth or large portions of it). He also dealt with measurement and representation of the Earth in time and space, attaching to it the gravitational field in a three-dimensional time-varying space (plane and non-plane).

It is because of his work as an engineering assistant for two years that he found a particular interest in geodesy. After engaging the railway constructions, he became an assistant in geodesy. He loved this aspect of mathematics so much that he became Professor of geodesy and practical geometry at the Technical University of Hanover in 1881 and until his death. His passion for mathematic was so extreme that he even wrote books on this specific subject. His best-known book was his *Handbuch der Vermessungskunde*, which means ''Textbook of Geodesy'', making three enormous volumes out of it.

Wilhelm Jordan is internationally known for his Gauss-Jordan elimination. Basically, what Jordan did was to improved the stability of the algorithm so it could be applied to minimize the squared error in the sum of a series of ''surveying observations''.

Source: historyoflinearalgebra.weebly.com/wilhelm-jordan-am.html