Mathematical induction

Prove that for positive integers n, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. (Hint: Use mathematical induction. That is, show the formula is true for $n = 1$, then assume it is true for n, and prove it is true for $n + 1$.) – this problem is from the 2016 Math Challenge Exam

For $n = 1$, we have: $1^2 = \frac{1(1+1)(2\cdot1+1)}{6}$, which is true.

Next, assume it is true for n and prove it is true for $n + 1$. We need to show that:

$$(1^2 + 2^2 + 3^2 + \cdots + n^2) + (n+1)^2 = \frac{(n+1)(n+1+1)(2(n+1)+1)}{6}$$

To do this, work each side down to a common expression:

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad = \frac{(n+1)(n+2)(2n+3)}{6}$$

Multiply both sides by 6:

$$n(n+1)(2n+1) + 6(n^2 + 2n + 1) = (n+1)(n+2)(2n+3)$$

$$2n^3 + 3n^2 + n + 6n^2 + 12n + 6 = (n+1)(2n^2 + 7n + 6)$$

$$2n^3 + 9n^2 + 13n + 6 = 2n^3 + 9n^2 + 13n + 6$$

QED

Try some of the following problems using mathematical induction:

1. $1 + 3 + 5 + \cdots + (2n-1) = n^2$
2. $n^3 + 2n$ is divisible by 3
3. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$
4. $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$
5. $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$
6. For $n \geq 4$, $n! > 2^n$
7. For $n \geq 2$, $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\ldots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$