Algebra
Determinants – The General Case

Determinants are very useful in matrix operations. The determinant of a **2 x 2 matrix** is defined to be:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

In words, the diagonals are multiplied and the product of second diagonal is subtracted from the product of the first diagonal. This process is generalized in determinants of larger matrices using what are referred to as minors. A minor is what is left of a determinant when the row and column of the element are eliminated.

The determinant of a matrix can be calculated by selecting a row and multiplying each element of the row by its corresponding minor. The results are alternately added and subtracted to get the value of the determinant. The ± sign of the each term is determined by the row and column in which it resides. The sign for the element in row m and column n is $(-1)^{(m+n)}$. The following matrices of signs show how they are applied to each row element:

2x2: $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ 3x3: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ 4x4: $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

Using minors of the first row to evaluate a **3 x 3 matrix**,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Or, using minors of the second column to evaluate the same **3 x 3 matrix**,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The results of the calculation will be the same, regardless of which row is selected, because of the power of matrices and determinants.

Example for a 3 x 3 matrix using minors of the first row:

$$\begin{vmatrix} 3 & 1 & 1 \\ 1 & -2 & -1 \\ -2 & 2 & 3 \end{vmatrix} = 3 \begin{vmatrix} -2 & -1 \\ 2 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix}$$

$$= 3(-4) - 1(1) + 1(-2) = \boxed{-15}$$

Note: this is the matrix that forms the denominator in the solution of the system of equations in the Cramer's Rule example.

The same process is followed for larger determinants. For example, a 5 x 5 determinant is first reduced to a sum of five elements each multiplied by their 4 x 4 minors. Each of the 4 x 4 minors is reduced to a sum of four elements each multiplied by their 3 x 3 minors, etc. The process is calculation intensive; today it would typically be performed using a computer.

Algebra
Cramer's Rule – 2 Equations

Cramer's Rule provides a powerful and simple way to solve systems of two or three linear equations. In larger systems of equations, it is a useful way to solve for just one of the variables, without having to solve the entire system of equations. To solve an entire system of four or more equations, a better technique would be Gauss-Jordan Elimination, especially if the student is aided by a computer and spreadsheet software such as Microsoft Excel.

Cramer's Rule works as long as the determinant of variable coefficients (i.e., the determinant in the denominator) is non-zero. If this determinant is zero, then there is no unique solution to the system of equations.

General Case for 2 Equations in 2 Unknowns

The standard form of the equations is:

$$a_1x + b_1y = k_1$$
$$a_2x + b_2y = k_2$$

Using determinant notation, Cramer's Rule states that the solutions for $\boldsymbol{x}$ and $\boldsymbol{y}$ are:

$$x = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Notice that the determinants in the denominators are the same; the columns in these determinants are the coefficients of the variables in the equations. The determinants in the numerators are almost the same as the ones in the denominators; the only difference is that the column of coefficients associated with the variable being evaluated is replaced by the equations' constant terms.

Example: Consider these equations:

$$3x - 6y = 18$$
$$x - 3y = 7$$

Then,

$$x = \frac{\begin{vmatrix} 18 & -6 \\ 7 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & -6 \\ 1 & -3 \end{vmatrix}} = \frac{-12}{-3} = \mathbf{4} \qquad y = \frac{\begin{vmatrix} 3 & 18 \\ 1 & 7 \end{vmatrix}}{\begin{vmatrix} 3 & -6 \\ 1 & -3 \end{vmatrix}} = \frac{3}{-3} = \mathbf{-1}$$

Algebra
Cramer's Rule – 3 Equations

General Case for 3 Equations in 3 Unknowns

The standard form of the equations is:

$$a_1x + b_1y + c_1z = k_1$$
$$a_2x + b_2y + c_2z = k_2$$
$$a_3x + b_3y + c_3z = k_3$$

Using determinant notation, Cramer's Rule states that the solutions for ***x, y*** and z are:

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

As in the case with two equations, the determinants in the denominators are all the same; the columns in these determinants are the coefficients of the variables in the equations. The determinants in the numerators are almost the same as the ones in the denominators; the only difference is that the column of coefficients associated with the variable being evaluated is replaced by the equations' constant terms.

Example: Consider these equations:

$$3x + y + z = 7$$
$$x - 2y - z = -2$$
$$-2x + 2y + 3z = -4$$

Note that the determinant of variable coefficients must be non-zero in order to use Cramer's Rule. If this determinant is zero, there is no unique solution to the system of equations.

Using determinant notation:

$$x = \frac{\begin{vmatrix} 7 & 1 & 1 \\ -2 & -2 & -1 \\ -4 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ 1 & -2 & -1 \\ -2 & 2 & 3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 3 & 7 & 1 \\ 1 & -2 & -1 \\ -2 & -4 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ 1 & -2 & -1 \\ -2 & 2 & 3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} 3 & 1 & 7 \\ 1 & -2 & -2 \\ -2 & 2 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ 1 & -2 & -1 \\ -2 & 2 & 3 \end{vmatrix}}$$

Performing the required calculations, we obtain the unique solution:

$$x = \frac{-30}{-15} = 2 \qquad y = \frac{-45}{-15} = 3 \qquad z = \frac{30}{-15} = -2$$