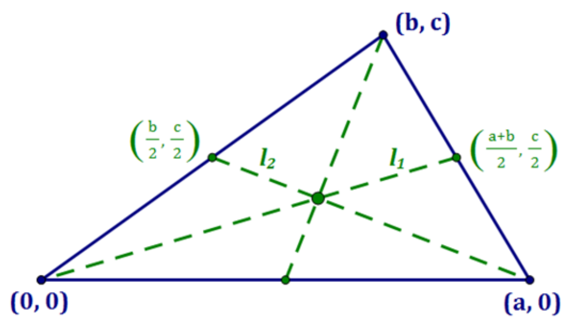
Euler Line

Challenge: Prove that the Centroid, Orthocenter and Circumcenter of a triangle lie on the same line. We will use Coordinate Geometry to do this by first determining the coordinates of each of these points and then showing that they lie on the same line.

In the material below, we refer to specific lines using the notations **"Line $\boldsymbol{i}$"** or "$\boldsymbol{l_i}$"; both forms refer to the "$\boldsymbol{i}$-th" line referenced in the material. In addition, we set the coordinates of the vertices of our triangle at $(0,0)$, $(a,0)$, and (b,c); this will provide a solution to the general case while facilitating our development.

Centroid (intersection of medians)



Line 1 contains the points $(0,0)$ and $\left(\frac{a+b}{2},\frac{c}{2}\right)$. Since the origin is on the line, its y-intercept is 0 and its slope is $m=\frac{\frac{c}{2}}{\frac{a+b}{2}}=\frac{c}{a+b}$. So, the equation of $\boldsymbol{l_1}$ is: $\boldsymbol{y=\frac{c}{a+b}x}$.

Line 2 contains the points $(a,0)$ and $\left(\frac{b}{2},\frac{c}{2}\right)$. Its slope is $m=\frac{\frac{c}{2}}{\frac{b}{2}-a}=\frac{c}{b-2a}$. Using point-slope form, the equation of $\boldsymbol{l_2}$ is: $\boldsymbol{y=\frac{c}{b-2a}(x-a)}$.

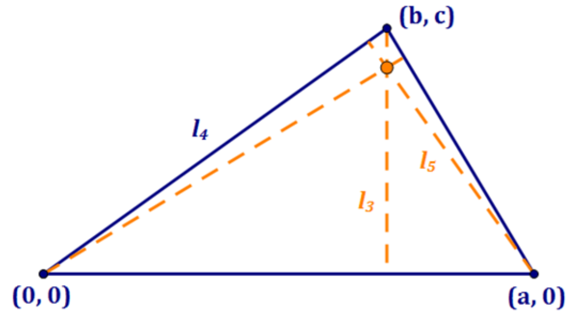
Intersection: Set the two equations equal in order to determine the point of intersection.

Set the equations equal:	$\frac{c}{a+b}x=\frac{c}{b-2a}(x-a)$
Cross multiply:	$c(b-2a)x=c(a+b)(x-a)$
Divide out c and distribute on the right:	$(b-2a)x=(a+b)x-a(a+b)$
Collect the x-terms:	$-3ax=-a(a+b)$
Divide by $-3a$:	$\boldsymbol{x=\frac{(a+b)}{3}}$
Substitute $x=\frac{(a+b)}{3}$ into l_1:	$\boldsymbol{y}=\frac{c}{a+b}x=\frac{c}{a+b}\cdot\frac{(a+b)}{3}=\boldsymbol{\frac{c}{3}}$

Centroid: The coordinates of the Centroid, then, are: $\boldsymbol{\left(\frac{a+b}{3},\frac{c}{3}\right)}$.

Orthocenter (intersection of altitudes)

Line 3 is a vertical line that contains the point (b, c). So, the equation of $\boldsymbol{l_3}$ is: $\boldsymbol{x = b}$.



Line 4 contains the points $(0, 0)$ and (b, c) so, its slope is $m = \frac{c}{b}$. **Line 5** is perpendicular to this, so its slope is $m = \frac{-b}{c}$. It also contains the point $(a, 0)$. Using point-slope form, the equation of $\boldsymbol{l_5}$ is: $\boldsymbol{y = \frac{-b}{c}(x - a)}$.

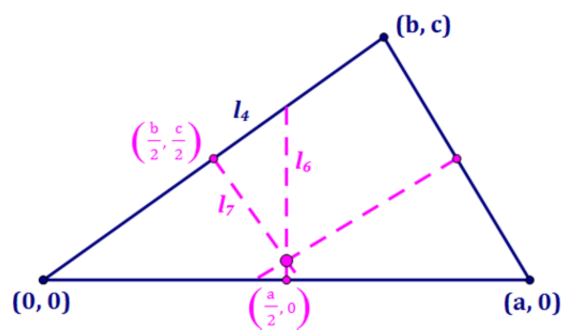
Intersection: Substitute $x = b$ into l_5 in order to determine the y-value of the point of intersection.

Substitute $x = b$ into l_5: $\quad y = \frac{-b}{c}(b - a) = \boldsymbol{\frac{-b^2 + ab}{c}}$

Orthocenter: The coordinates of the Orthocenter, then, are: $\boldsymbol{\left(b, \frac{-b^2 + ab}{c}\right)}$.

Circumcenter (intersection of perpendicular bisectors)

Line 6 is a vertical line that contains the point $\left(\frac{a}{2}, 0\right)$. So, the equation of $\boldsymbol{l_6}$ is: $\boldsymbol{x = \frac{a}{2}}$.



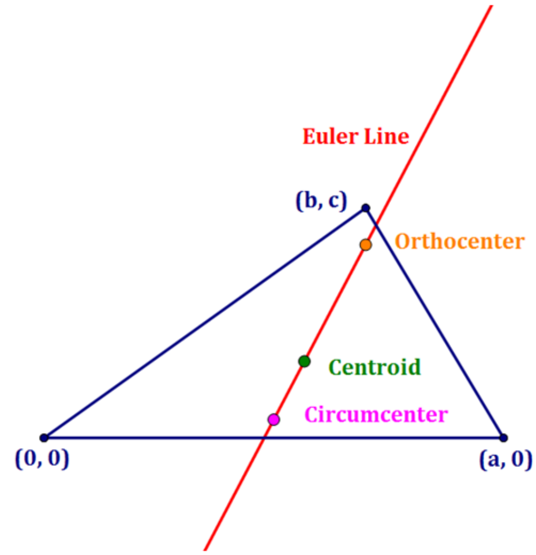
Line 7, like **Line 5** (above), is perpendicular to **Line 4**, so its slope is $m = \frac{-b}{c}$. It also contains the point $\left(\frac{b}{2}, \frac{c}{2}\right)$. Using point-slope form, the equation of $\boldsymbol{l_7}$ is: $\boldsymbol{y = \frac{-b}{c}\left(x - \frac{b}{2}\right) + \frac{c}{2}}$.

Intersection: Substitute $x = \frac{a}{2}$ into l_7 in order to determine the y-value of the point of intersection.

Substitute $x = \frac{a}{2}$ into l_7: $\quad y = \frac{-b}{c}\left(\frac{a}{2} - \frac{b}{2}\right) + \frac{c}{2} = \boldsymbol{\frac{b^2 - ab + c^2}{2c}}$

Circumcenter: The coordinates of the Circumcenter, then, are: $\boldsymbol{\left(\frac{a}{2}, \frac{b^2 - ab + c^2}{2c}\right)}$.

The Euler Line



We will now prove that the Centroid, Orthocenter and Circumcenter of a triangle lie on the same line by first determining the equation of the line that contains the Centroid and Orthocenter, and then showing that the Circumcenter lies on that line.

Equation of the Euler Line

The slope of the line that contains the **Centroid** $\left(\frac{a+b}{3}, \frac{c}{3}\right)$ and **Orthocenter** $\left(b, \frac{-b^2+ab}{c}\right)$ is:

$$m = \frac{\frac{-b^2+ab}{c} - \frac{c}{3}}{b - \frac{a+b}{3}} = \frac{-3b^2+3ab-c^2}{c(2b-a)}$$

Using point-slope form, with the point being the **Centroid** $\left(\frac{a+b}{3}, \frac{c}{3}\right)$, the equation of the **Euler Line** is:

$$y = \frac{-3b^2+3ab-c^2}{c(2b-a)}\left(x - \frac{a+b}{3}\right) + \frac{c}{3}$$

Now, let's substitute the x-value of the **Circumcenter** $\left(\frac{a}{2}\right)$ into this equation and see if this yields the y-value of the **Circumcenter** $\left(\frac{b^2-ab+c^2}{2c}\right)$. If it does, the **Circumcenter** lies on the same line, and we have completed our proof.

$$y = \frac{-3b^2+3ab-c^2}{c(2b-a)}\left(\frac{a}{2} - \frac{a+b}{3}\right) + \frac{c}{3}$$

$$= \frac{-3b^2+3ab-c^2}{c(2b-a)}\left(\frac{a-2b}{6}\right) + \frac{c}{3}$$

$$= \frac{-(-3b^2+3ab-c^2)}{6c} + \frac{c}{3}$$

$$= \frac{3b^2-3ab+c^2}{6c} + \frac{2c^2}{6c}$$

$$= \frac{3b^2-3ab+3c^2}{6c} = \frac{b^2-ab+c^2}{2c}$$

Since this is the y-value of the **Circumcenter,** we have completed our proof.