CALCULUS AB
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is measured in hours and $0 \le t \le 8$. At the beginning of the workday $(t = 0)$, the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \le t \le 8$, the plant processes gravel at a constant rate of 100 tons per hour.

Let's start by summarizing a few things:

Rate of input (gravel arriving): $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right),\ 0 \le t \le 8$

Rate of output (gravel processed): $F(t) = 100,\ 0 \le t \le 8$

Rate of change of amount of gravel: $H(t) = G(t) - F(t)$, so

$H(t) = 45\cos\left(\frac{t^2}{18}\right) - 10,\ 0 \le t \le 8$

Starting amount of unprocessed gravel: 500 tons

(a) Find $G'(5)$. Using correct units, interpret your answer in the context of the problem.

$$G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$$

$$G'(t) = -45\sin\left(\frac{t^2}{18}\right)\cdot\frac{2t}{18} = -5t\sin\left(\frac{t^2}{18}\right)$$

$$G'(5) = -5\cdot 5\sin\left(\frac{5^2}{18}\right) = -25\sin\left(\frac{25}{18}\right) = \mathbf{-24.588}$$ (remember to use radians)

The **units** for $G(t)$ are tons per hour, so the units for $G'(t)$ are tons per hour per hour, or $\frac{\textbf{tons}}{\textbf{hour}^2}$

Interpretation: the acceleration in arriving gravel at time $t = 5$ hours is $-24.588\ \frac{\text{tons}}{\text{hour}^2}$. Alternatively, the rate at which gravel is arriving is slowing down at $-24.588\ \frac{\text{tons}}{\text{hour}^2}$ at time $t = 5$ hours.

(b) Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday.

Unprocessed gravel arriving is based on $G(t)$, which is the rate at which gravel arrives.

$$\int_0^8 G(t)\,dt = \int_0^8 \left[90 + 45\cos\left(\frac{t^2}{18}\right)\right] dt = \mathbf{825.551} \qquad \text{(using calculator)}$$

(c) Is the amount of unprocessed gravel at the plant increasing or decreasing at time $t = 5$ hours? Show the work that leads to your answer.

The amount of unprocessed gravel is based on the function $H(t)$ that we defined above.

$H(t)$ is the rate at which the amount unprocessed gravel is changing, so we can look at $H(5)$ to answer this question.

$$H(t) = 45\cos\left(\frac{t^2}{18}\right) - 10$$

$$H(5) = 45\cos\left(\frac{5^2}{18}\right) - 10 = \mathbf{-1.859}\ \frac{\textbf{tons}}{\textbf{hour}} \qquad \text{(remember to use radians)}$$

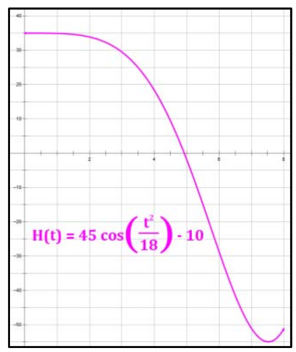
So the amount of unprocessed gravel at time $t = 5$ is **decreasing.**

(d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer.

A maximum must occur either at the endpoints of the period or at a critical value in the period. Let's find the critical value(s) of t in the interval $0 \le t \le 8$. Recall that $H(t)$ is the rate of change in the amount of gravel. A critical value is obtained when $H(t) = 0$.

$$H(t) = 45\cos\left(\frac{t^2}{18}\right) - 10 = 0$$

$$t = \sqrt{18\cos^{-1}\left(\frac{2}{9}\right)} = 4.923 \text{ hours}$$



Note that the graph of $H(t)$ at right confirms that this is indeed a point where $H(t) = 0$. Note also that $H(t) > 0$ to the left of $t = 4.923$ and that $H(t) < 0$ to the right of $t = 4.923$. This proves that $t = 4.923$ is a relative maximum.

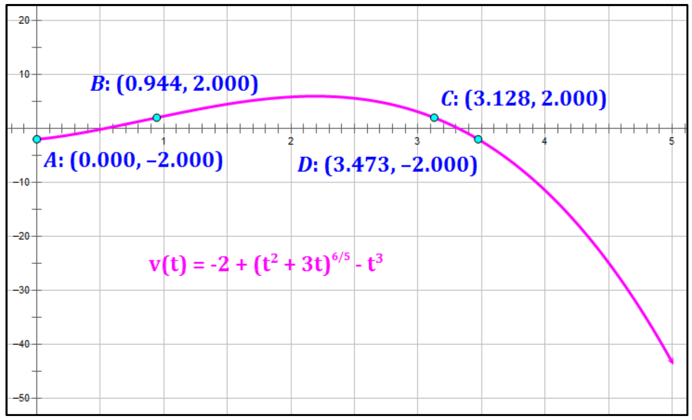
Further, because $H(t) > 0$ for all $t < 4.923$, we can conclude that the amount of unprocessed gravel is lower at $t = 0$ than it is at $t < 4.923$. Likewise, because $H(t) < 0$ for all $t > 4.923$, we can conclude that the amount of unprocessed gravel is lower at $t = 8$ than it is at $t = 4.923$. So the global maximum occurs at $t = 4.923$. Finally,

$$\text{Amount of Gravel } (4.923) = 500 + \int_0^{4.923} \left[45\cos\left(\frac{t^2}{18}\right) - 10\right] dt = \mathbf{635.376}$$

2. A particle moves along a straight line. For $0 \le t \le 5$, the velocity of the particle is given by $v(t) = -2 + \left(t^2 + 3t\right)^{6/5} - t^3$, and the position of the particle is given by $s(t)$. It is known that $s(0) = 10$.

(a) Find all values of t in the interval $2 \le t \le 4$ for which the speed of the particle is 2.

Recall that speed is the absolute value of velocity, so we want all values of t where $v(t) = 2$ and where $v(t) = -2$. Use a calculator to graph the function and find these values.

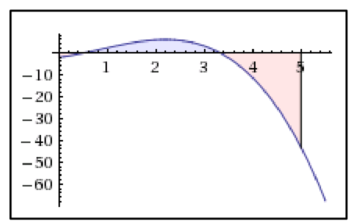


Based on the graph, we conclude that there are two points in the interval $[2, 4]$ where speed $= 2$, namely:

$$\boldsymbol{t = \{3.128, 3.473\}}$$

(b) Write an expression involving an integral that gives the position $s(t)$. Use this expression to find the position of the particle at time $t = 5$.

$$s(t) = s(0) + \int_0^t v(x)\,dx = \boldsymbol{s(0) + \int_0^t \left[-2 + (x^2 + 3x)^{6/5} - x^3\right] dx}$$

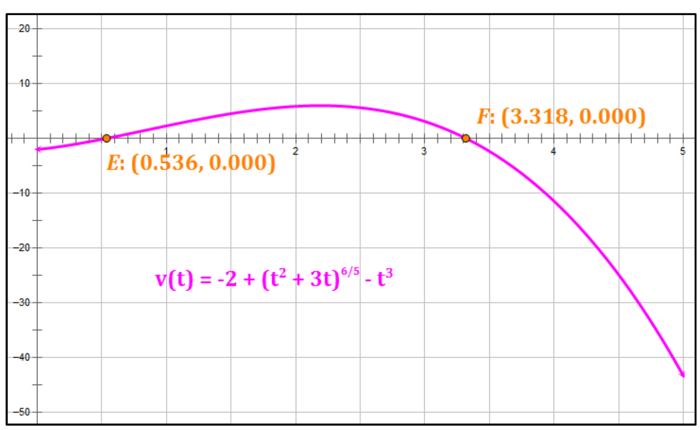


$$s(5) = s(0) + \int_0^5 \left[-2 + (x^2 + 3x)^{6/5} - x^3\right] dx$$

$$= 10 - 19.207 = \boldsymbol{-9.207}$$

(c) Find all times t in the interval $0 \le t \le 5$ at which the particle changes direction. Justify your answer.

$$v(t) = -2 + (x^2 + 3x)^{6/5} - x^3$$



The particle changes direction when velocity changes from positive to negative or from negative to positive. Based on the graph of the function, this occurs at two values of t:

$$\boldsymbol{t = \{0.536, 3.318\}}$$

(d) Is the speed of the particle increasing or decreasing at time $t = 4$? Give a reason for your answer.

Speed is increasing when the signs of the first and second derivatives of the position function (i.e., velocity and acceleration) are the same. Speed is decreasing when the signs of the first and second derivatives of the position function are different.

Velocity:

$$v(t) = -2 + (t^2 + 3t)^{6/5} - t^3$$

$$v(4) = -2 + (4^2 + 3 \cdot 4)^{6/5} - 4^3 = \mathbf{-11.476}$$

Acceleration:

$$a(t) = v'(t) = \frac{6}{5}(t^2 + 3t)^{1/5}(2t + 3) - 3t^2$$

$$a(4) = v'(4) = \frac{6}{5}(4^2 + 3 \cdot 4)^{1/5}(2 \cdot 4 + 3) - 3 \cdot 4^2 = \mathbf{-22.296}$$

Since velocity and acceleration are both negative (i.e., they have the same sign) at $t = 4$, we conclude that speed is **increasing** at $t = 4$.

CALCULUS AB
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

3. Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

$C'(3.5)$ can be estimated as the slope of the secant line containing $C(3)$ and $C(4)$.

$$C'(3.5) \sim \frac{C(4) - C(3)}{4 - 3} = \frac{12.8 - 11.2}{1} = \mathbf{1.6\ ounces\ per\ minute}$$

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

The Mean Value Theorem states that if a function f is continuous on an interval $[a, b]$ and differentiable on the interval (a, b), then there is at least one value c such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

For this problem, let $a = 2, b = 4$. The function C is differentiable on the interval $(0, 6)$ and is, therefore, differentiable on the interval $(2, 4)$ and continuous on the interval $[2, 4]$. It follows, then, that there is at least one value t such that:

$$C'(t) = \frac{C(b) - C(a)}{b - a} = \frac{C(4) - C(2)}{4 - 2} = \frac{12.8 - 8.8}{4 - 2} = 2$$

Therefore, **there is a time t such that $C'(t) = 2$.**

(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

To calculate the midpoint sum with three subintervals of equal length, the length of each sub-interval must be $6 \div 3 = 2$, so the intervals are $[0, 2], [2, 4]$ and $[4, 6]$. The midpoints of these intervals are $t = \{1, 3, 5\}$. Then,

$$\int_0^6 C(t)\,dt = 2 \cdot (5.3 + 11.2 + 13.8) = 60.6 \text{ ounces}$$

$$\mathbf{\frac{1}{6}\int_0^6 C(t)\,dt = 10.1 \text{ ounces}}$$

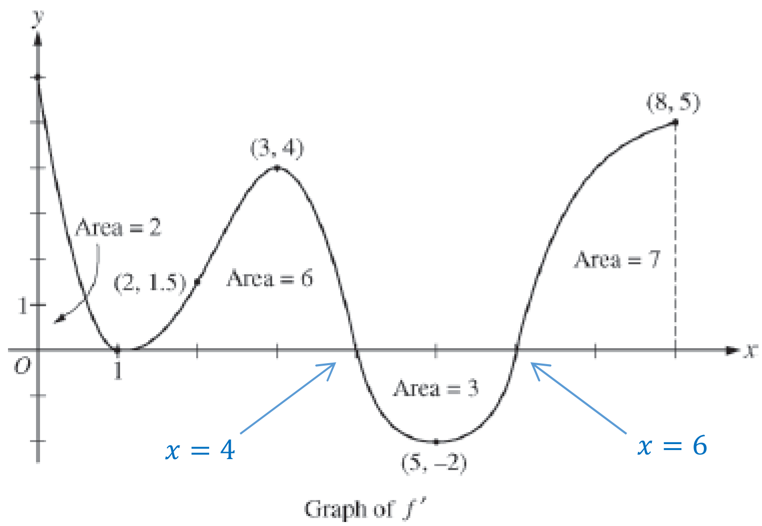
Explanation: $\int_0^6 C(t)\,dt$ is the total number of ounces of coffee that drips into the large cup over the 6 minute interval being measured. **So, $\frac{1}{6}\int_0^6 C(t)\,dt$ is the average number of ounces of coffee that drips into the large cup in one minute over the 6 minute interval being measured.**

(d) The amount of coffee in the cup, in ounces, is modeled by $B(t) = 16 - 16e^{-0.4t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t = 5$.

$$B(t) = 16 - 16e^{-0.4t}$$

$$B'(t) = (-0.4)(-16e^{-0.4t}) = 6.4\,e^{-0.4t}$$

$$B'(5) = 6.4\,e^{-0.4\cdot 5} = \mathbf{6.4\,e^{-2} = \frac{6.4}{e^2} \text{ ounces per minute}}$$



Graph of f'

4. The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

f has a local minimum at $x = a$ if and only if $f'(a) = 0$ and f' changes from negative to positive as the curve passes through $x = a$. This occurs only at $\boldsymbol{x = 6}$.

(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

The absolute minimum occurs at either an endpoint or a local minimum. So let's test each.

Endpoint: $f(8) = 4$

Local min: $f(6) = f(8) - \int_6^8 f(x)\,dx = 4 - 7 = -3$

Endpoint: $f(0) = f(6) - \int_0^6 f(x)\,dx = -3 - (-3 + 6 + 2) = -8$

So, the absolute minimum occurs at $x = 0$, and the absolute minimum value of f is – **8.**

(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

f is concave down where the curve of f' is decreasing.

f is increasing where the curve of f' is positive.

These conditions are met simultaneously on the open intervals $\boldsymbol{x \in \{(0, 1), (3, 4)\}}$.

(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

The slope of the tangent line at $x = 3$ is $g'(3)$.

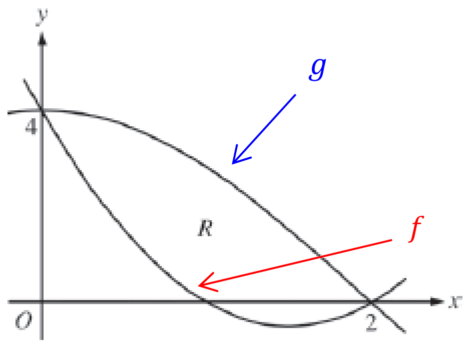
$g(x) = (f(x))^3$

$f'(3) = 4$ from the above graph.

$g'(x) = 3\,(f(x))^2 \cdot f'(x)$

$g'(3) = 3\,(f(3))^2 \cdot f'(3) = 3 \cdot \left(-\frac{5}{2}\right)^2 \cdot 4$

$= \mathbf{75}$



5. Let $f(x) = 2x^2 - 6x + 4$ and $g(x) = 4\cos\left(\frac{1}{4}\pi x\right)$. Let R be the region bounded by the graphs of f and g, as shown in the figure above.

(a) Find the area of R.

First notice that the top function is: $g(x) = 4\,cos\left(\frac{1}{4}\pi x\right)$ and the bottom function is: $f(x) = 2x^2 - 6x + 4$. Then,

$$A = \int_0^2 [g(x) - f(x)]\,dx = \int_0^2 \left[4\,cos\left(\frac{1}{4}\pi x\right) - (2x^2 - 6x + 4)\right] dx$$

$$= 4\int_0^2 cos\left(\frac{1}{4}\pi x\right) dx - \int_0^2 (2x^2 - 6x + 4)\,dx$$

Next, use u-substitution on the first integral.

Let: $u = \frac{1}{4}\pi x$. Then: $du = \frac{1}{4}\pi\,dx$ and so, $\frac{4}{\pi}\,du = dx$.

When $x = 0,\ u = 0$ and when $x = 2,\ u = \frac{\pi}{2}$. Now, back to calculating the area.

$$A = 4\int_0^{\frac{\pi}{2}} (\cos u)\cdot\frac{4}{\pi}\,du \ - \int_0^2 (2x^2 - 6x + 4)\,dx$$

$$= \frac{16}{\pi}\sin u\,\Big|_0^{\pi/2} \ - \ \left(\frac{2}{3}x^3 - 3x^2 + 4x\right)\Big|_0^2$$

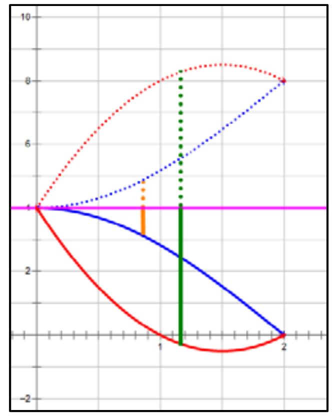
$$= \left[\frac{16}{\pi}(1-0)\right] \ - \ \left[\left(\frac{2}{3}\cdot 2^3 - 3\cdot 2^2 + 4\cdot 2\right) - 0\right]$$

$$= \frac{16}{\pi} - \left(\frac{2}{3}\cdot 2^3 - 3\cdot 2^2 + 4\cdot 2\right) = \frac{16}{\pi} - \left(\frac{16}{3} - 12 + 8\right)$$

$$= \boldsymbol{\frac{16}{\pi} - \frac{4}{3} = \frac{48 - 4\pi}{3\pi}}$$

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 4$.

We must use the **washer method** because there is a gap between the region we are revolving and its reflection across the axis of revolution, $\boldsymbol{y = 4}$.



The washer method is simply a double use of the disk method. We create a large disk and a small disk, and then subtract the two to obtain the "washer."

Revolving about a horizontal line means our **disks are vertical**, as shown, and we should **integrate with respect to $\boldsymbol{x}$**.

The height of the **larger disk** is the distance between the axis of revolution, $y = 4$ and the curve $y = 2x^2 - 6x + 4$, so the height is $f(x) = 4 - (2x^2 - 6x + 4) = -2x^2 + 6x$.

The height of the **smaller disk** is the distance between the axis of revolution, $y = 4$ and the curve $y = 4\cos\left(\frac{1}{4}\pi x\right)$, so the height is:
$g(x) = 4 - 4\cos\left(\frac{1}{4}\pi x\right)$.

We move the disks from left to right through the region, i.e., from $x = 0$ to $x = 2$.

The volume is determined from the formula: $V = \pi \int_a^b \left[(f(x))^2 - \left(g(x)\right)^2\right] dx$.
Therefore,

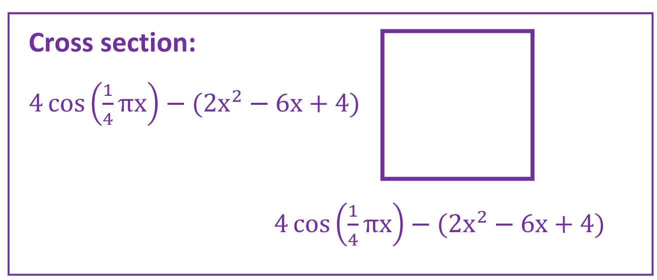
$$\boldsymbol{V = \pi \int_0^2 \left[(-2x^2 + 6x)^2 - \left(4 - 4\cos\left(\frac{1}{4}\pi x\right)\right)^2\right] dx}$$

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Write, but do not evaluate, an integral expression that gives the volume of the solid.

The x-values at the endpoints of the region are $\boldsymbol{x = \{0, 2\}}$.

The difference of the curves is: $4\cos\left(\frac{1}{4}\pi x\right) - (2x^2 - 6x + 4)$.

The formula for the area of a square is $A = s^2$, where s is the length of a side of the square.



Since the cross sections are squares, we set: $s = 4\cos\left(\frac{1}{4}\pi x\right) - (2x^2 - 6x + 4)$, apply the formula for the area of a square and integrate to get the volume:

$$V = \int_0^2 A(x)\,dx = \int_0^2 \boldsymbol{\left(4\cos\left(\frac{1}{4}\pi x\right) - (2x^2 - 6x + 4)\right)^2 dx}$$

6. Consider the differential equation $\frac{dy}{dx} = e^y(3x^2 - 6x)$. Let $y = f(x)$ be the particular solution to the differential equation that passes through $(1, 0)$.

(a) Write an equation for the line tangent to the graph of f at the point $(1, 0)$. Use the tangent line to approximate $f(1.2)$.

$$\frac{dy}{dx} = e^y\,(3x^2 - 6x)$$

At the point $(1, 0)$: $\frac{dy}{dx} = e^y\,(3x^2 - 6x) = e^0\,(3 \cdot 1^2 - 6 \cdot 1) = -3$

So, the slope of the line is $m = -3$. Recall that a point on the line is $(1, 0)$. Then, an equation for the tangent line is:

$\boldsymbol{y = -3\,(x - 1)}$ or $\boldsymbol{y = -3x + 3}$

Then,

$\boldsymbol{f(1.2)} = -3(1.2) + 3 = \boldsymbol{-0.6}$

(b) Find $y = f(x)$, the particular solution to the differential equation that passes through $(1, 0)$.

$$\frac{dy}{dx} = e^y\,(3x^2 - 6x)$$

$$\frac{dy}{e^y} = (3x^2 - 6x)\,dx \quad \text{so,} \quad e^{-y}\,dy = (3x^2 - 6x)\,dx$$

$$\int e^{-y}\,dy = \int (3x^2 - 6x)\,dx$$

$$-e^{-y} = x^3 - 3x^2 + C$$

Substituting in $(x, y) = (1, 0)$ gives: $-e^0 = 1^3 - 3 \cdot 1^2 + C$ or $-1 = -2 + C$ so, $C = 1$

Substituting $C = 1$ and continuing with the Algebra gives:

$$-e^{-y} = x^3 - 3x^2 + 1$$

$$e^{-y} = -x^3 + 3x^2 - 1$$

$$-y = \ln(-x^3 + 3x^2 - 1)$$

$$\boldsymbol{y = -\ln(-x^3 + 3x^2 - 1)}$$