Applications of Integration – Challenge Problems:

Learning Calculus for the AP exam is not just about learning what you are taught in class and doing problems to demonstrate that knowledge. It's about applying that knowledge to situations you may not have seen before.

Toward that end, here are a couple of problems that I hope you will find to be entertaining and fun, and from which you may learn something. Stretching your mind this way is also good practice for the Mathematical Challenge Scholarship Exam in May.

The problems are presented on this page. The next page contains the solutions. Enjoy!

1) Let $f(x)$ be an odd function. What is $\int_{-a}^{a} f(x)\,dx$? Explain.

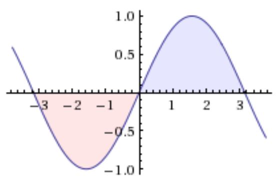
   (Hint: an odd function is one for which $f(-x) = -f(x)$. Examples are: $f(x) = \sin x$, $f(x) = x$, $f(x) = x^3$. Look at the graphs of a couple of these and see if you can reason out the solution.)

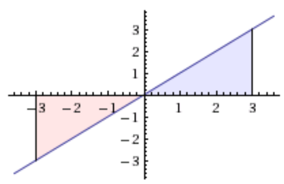
2) Solve: $\int_0^\infty \frac{dx}{1+e^{ax}}$.

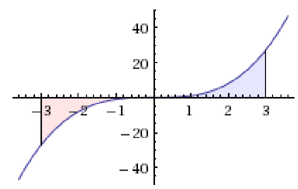
Solutions:

1) Let $f(x)$ be an odd function. What is $\int_{-a}^{a} f(x)\,dx$? Explain. (Hint: an odd function is one for which $f(-x) = -f(x)$. Examples are: $f(x) = \sin x,\ f(x) = x,\ f(x) = x^3$. Look at the graphs of a couple of these and see if you can reason out the solution.)

Looking at graphs of $f(x) = \sin x,\ f(x) = x,\ f(x) = x^3$ over an interval $[-a, a]$, we notice that the area between the curve and the x-axis that is to the left of the y-axis is equal to but opposite in sign from the area between the curve and the x-axis that is to the right of the y-axis.







Therefore the two areas will add to zero. That is, **For an odd function, $f(x)$, $\int_{-a}^{a} f(x)\,dx = 0$.**

2) Solve: $\int_0^\infty \frac{dx}{1+e^{ax}}$ for $a > 0$.

Let: $u = e^{ax}$, $du = ae^{ax}dx$. Then, $dx = \dfrac{du}{ae^{ax}} = \dfrac{du}{au}$.

When $x = 0,\ u = 1$ and when $x = \infty,\ u = \infty$.

$$\int_0^\infty \frac{1}{1+e^{ax}}dx = \int_1^\infty \frac{1}{1+u}\cdot\frac{du}{au} = \frac{1}{a}\int_1^\infty \frac{1}{u(1+u)}du$$

Next, split $\dfrac{1}{u(1+u)}$ into separate fractions: $\dfrac{1}{u(1+u)} = \dfrac{1}{u} - \dfrac{1}{u+1}$. Continuing,

$$\frac{1}{a}\int_1^\infty \frac{1}{u(1+u)}du = \frac{1}{a}\int_1^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right)du = \frac{1}{a}[\ln u - \ln(u+1)]\Big|_1^\infty$$

$$= \frac{1}{a}\ln\left(\frac{u}{u+1}\right)\Big|_1^\infty = \frac{1}{a}\left(\ln 1 - \ln\frac{1}{2}\right)$$

$$= \frac{1}{a}(0 + \ln 2) = \frac{\ln 2}{a}$$

Therefore, we have the general solution:

$$\boldsymbol{\int_0^\infty \frac{1}{1+e^{ax}}dx = \frac{\ln 2}{a}}$$

Some specific solutions (i.e., with specific values of a) would be:

$$\int_0^\infty \frac{1}{1+e^{2x}}dx = \frac{\ln 2}{2},\quad \int_0^\infty \frac{1}{1+e^{5x}}dx = \frac{\ln 2}{5},\quad \int_0^\infty \frac{1}{1+e^{x/2}}dx = \frac{\ln 2}{\frac{1}{2}} = 2\ln 2 = \ln 4$$