Algebra
Introduction to Polynomials

What is a Polynomial?

A polynomial is an expression that can be written as a term or a sum of terms, each of which is the product of a scalar (the coefficient) and a series of variables. Each of the terms is also called a monomial.

Examples (all of these are polynomials):

Monomial	$3x$	$-4x^{12}y^3$
Binomial	$2x+8$	$15xyz^7-12xyz$
Trinomial	x^2-6x+9	$x^5+7x^4-3x^3$
Other	$x^4-4x^3+6x^2-4x+1$	$-2xyz+6xz^2+3yz-8xz+2z^5$

Definitions:

Scalar: A real number.

Monomial: Polynomial with one term.

Binomial: Polynomial with two terms.

Trinomial: Polynomial with three terms.

Degree of a Polynomial

The degree of a monomial is the sum of the exponents on its variables.

The degree of a polynomial is the highest degree of any of its monomial terms.

Examples:

Polynomial	**Degree**
6	0
$3x$	1
$3xyz$	3

Polynomial	**Degree**
$3x^3yz+6x^2yz^3$	6
$15xyz^7-12xyz$	9
$x^5+7x^4-3x^3$	5

Algebra
Adding and Subtracting Polynomials

Problems asking the student to add or subtract polynomials are often written in linear form:

$$\text{Add: } (3x^3 + 2x - 4) + (-2x^2 + 4x + 6)$$

The problem is much more easily solved if the problem is written in column form, with each polynomial written in standard form.

Definitions

Standard Form: A polynomial in standard form has its terms written from highest degree to lowest degree from left to right.

Example: The standard form of $(x + 3x^2 + 4)$ is $(3x^2 + x + 4)$

Like Terms: Terms with the same variables raised to the same powers. Only the numerical coefficients are different.

Example: $2xz^5$, $-6xz^5$, and xz^5 are like terms.

Addition and Subtraction Steps

Step 1: Write each polynomial in standard form. Leave blank spaces for missing terms. For example, if adding $(3x^3 + 2x - 4)$, leave space for the missing x^2-term.

Step 2: If you are subtracting, change the sign of each term of the polynomial to be subtracted and add instead. Adding is much easier than subtracting.

Step 3: Place the polynomials in column form, being careful to line up like terms.

Step 4: Add the polynomials.

Examples:

Add: $(3x^3 + 2x - 4) + (-2x^2 + 4x + 6)$

Solution:

$$\begin{array}{rrrrr} & 3x^3 & & +\,2x & -\,4 \\ + & & -\,2x^2 & +\,4x & +\,6 \\ \hline & 3x^3 & -\,2x^2 & +\,6x & +\,2 \end{array}$$

Subtract: $(3x^3 + 2x - 4) - (-2x^2 + 4x + 6)$

Solution:

$$\begin{array}{rrrrr} & 3x^3 & & +\,2x & -\,4 \\ + & & 2x^2 & -\,4x & -\,6 \\ \hline & 3x^3 & +\,2x^2 & -\,2x & -\,10 \end{array}$$

Algebra
Multiplying Binomials

The three methods shown below are equivalent. Use whichever one you like best.

FOIL Method

FOIL stands for First, Outside, Inside, Last. To multiply using the FOIL method, you make four separate multiplications and add the results.

Example: Multiply $(2x+3)\cdot(3x-4)$

First: $2x\cdot 3x = 6x^2$

Outside: $2x\cdot(-4) = -8x$

Inside: $3\cdot(3x) = 9x$

Last: $3\cdot(-4) = -12$

The result is obtained by adding the results of the 4 separate multiplications.

F O I L

$$(2x+3)\cdot(3x-4) = 6x^2 - 8x + 9x - 12$$
$$= 6x^2 + x - 12$$

Box Method

The Box Method is pretty much the same as the FOIL method, except for the presentation. In the box method, a 2x2 array of multiplications is created, the 4 multiplications are performed, and the results are added.

Example: Multiply $(2x+3)\cdot(3x-4)$

Multiply	**3x**	**−4**
2x	$6x^2$	$-8x$
+3	$9x$	-12

The result is obtained by adding the results of the 4 separate multiplications.

$$(2x+3)\cdot(3x-4) = 6x^2 - 8x + 9x - 12$$
$$= 6x^2 + x - 12$$

Stacked Polynomial Method

A third method is to multiply the binomials like you would multiply 2-digit numbers. The name comes from how the two polynomials are placed in a "stack" in preparation for multiplication.

Example: Multiply $(2x+3)\cdot(3x-4)$

$$\begin{array}{r} (2x+3) \\ \cdot\ \ (3x-4) \\ \hline -8x-12 \\ 6x^2+9x \\ \hline 6x^2+\ x-12 \end{array}$$

Algebra
Multiplying Polynomials

If the polynomials to be multiplied contain more than two terms (i.e., they are larger than binomials), the FOIL Method will not work. Instead, either the Box Method or the Stacked Polynomial Method should be used. Notice that each of these methods is essentially a way to apply the distributive property of multiplication over addition.

The methods shown below are equivalent. Use whichever one you like best.

Box Method

The Box Method is the same for larger polynomials as it is for binomials, except the box is bigger. An array of multiplications is created; the multiplications are performed; and like terms are added.

Example: Multiply $(x^3 - 2x^2 + 2x + 3) \cdot (2x^2 - 3x - 4)$

Multiply	$\mathbf{2x^2}$	$\mathbf{-3x}$	$\mathbf{-4}$
$\mathbf{x^3}$	$2x^5$	$-3x^4$	$-4x^3$
$\mathbf{-2x^2}$	$-4x^4$	$+6x^3$	$+8x^2$
$\mathbf{+2x}$	$+4x^3$	$-6x^2$	$-8x$
$\mathbf{+3}$	$+6x^2$	$-9x$	-12

Results:

$$(x^3 - 2x^2 + 2x + 3) \cdot (2x^2 - 3x - 4)$$

$$= \quad 2x^5$$

$$-4x^4 - 3x^4$$

$$+4x^3 + 6x^3 - 4x^3$$

$$+6x^2 - 6x^2 + 8x^2$$

$$-9x - 8x$$

Stacked Polynomial Method

In the Stacked Polynomial Method, the polynomials are multiplied using the same technique to multiply multi-digit numbers One helpful tip is to place the smaller polynomial below the larger one in the stack.

Results:

$$\begin{array}{r}
x^3 - 2x^2 + 2x + 3 \\
\cdot \quad 2x^2 - 3x - 4 \\
\hline
-4x^3 + 8x^2 - 8x - 12 \\
-3x^4 + 6x^3 - 6x^2 - 9x \quad\quad \\
2x^5 - 4x^4 + 4x^3 + 6x^2 \quad\quad\quad\quad \\
\hline
2x^5 - 7x^4 + 6x^3 + 8x^2 - 17x - 12
\end{array}$$

Algebra
Dividing Polynomials

Dividing polynomials is performed much like dividing large numbers long-hand.

Long Division Method

This process is best described by example:

Example: $(2x^3 + 5x^2 + x - 2) \div (x + 2)$

Step 1: Set up the division like a typical long hand division problem.

$$\begin{array}{rl} x+2 & \big)\overline{\,2x^3+5x^2+x-2\,} \end{array}$$

Step 2: Divide the leading term of the dividend by the leading term of the divisor. Place the result above the term of like degree of the dividend.

$(2x^3) \div x = 2x^2$

$$\begin{array}{rl} & \,2x^2 \\ x+2 & \big)\overline{\,2x^3+5x^2+x-2\,} \end{array}$$

Step 3: Multiply the new term on top by the divisor and subtract from the dividend.

$$(2x^2)(x+2) = 2x^3 + 4x^2$$

$$\begin{array}{rl} & \,2x^2 \\ x+2 & \big)\overline{\,2x^3+5x^2+x-2\,} \\ & \,\underline{2x^3+4x^2} \\ & x^2+x-2 \end{array}$$

Step 4: Repeat steps 2 and 3 on the remainder of the division until the problem is completed.

$$\begin{array}{rl} & \,2x^2+x-1 \\ x+2 & \big)\overline{\,2x^3+5x^2+x-2\,} \\ & \,\underline{2x^3+4x^2} \\ & x^2+x-2 \\ & \underline{x^2+2x} \\ & -x-2 \\ & \underline{-x-2} \\ & 0 \end{array}$$

This process results in the final answer appearing above the dividend, so that:

$(2x^3 + 5x^2 + x - 2) \div (x + 2) = 2x^2 + x - 1$

Remainders

If there were a remainder, it would be appended to the result of the problem in the form of a fraction, just like when dividing integers. For example, in the problem above, if the remainder were 3, the fraction $\frac{3}{x+2}$ would be added to the result of the division. $(2x^3 + 5x^2 + x + 1) \div (x + 2) = 2x^2 + x - 1 + \frac{3}{x+2}$

Alternatives

This process can be tedious. Fortunately, there are better methods for dividing polynomials than long division. These include **Factoring**, which is discussed next and elsewhere in this Guide, and **Synthetic Division**, which is discussed in the chapter on Polynomials – Intermediate.

Algebra
Factoring Polynomials

Polynomials cannot be divided in the same way numbers can. In order to divide polynomials, it is often useful to factor them first. Factoring involves extracting simpler terms from the more complex polynomial.

Greatest Common Factor

The Greatest Common Factor of the terms of a polynomial is determined as follows:

Step 1: Find the Greatest Common Factor of the coefficients.

Step 2: Find the Greatest Common Factor for each variable. This is simply each variable taken to the lowest power that exists for that variable in any of the terms.

Step 3: Multiply the GCF of the coefficients by the GCF for each variable.

Example:

Find the GCF of $(18x^5y^6z + 42x^3y^7z^3 + 30x^8z^6)$

The GCF of the coefficients and each variable are shown in the box to the right. The GCF of the polynomial is the product of the four individual GCFs.

$\text{GCF } (18, 42, 30) = 6$

$\text{GCF } (x^5, x^3, x^8) = x^3$

$\text{GCF } (y^6, y^7, 1) = 1$

$\text{GCF } (z,\ z^3, z^6) = z$

So, $\text{GCF (polynomial)} = 6x^3z$

Factoring Steps

Step 1: Factor out of all terms the GCF of the polynomial.

Step 2: Factor out of the remaining polynomial any binomials that can be extracted.

Step 3: Factor out of the remaining polynomial any trinomials that can be extracted.

Step 4: Continue this process until no further simplification is possible.

Note: Typically only steps 1 and 2 are needed in high school algebra problems.

Examples:

Factor: $3x^4y - 18x^3y + 27x^2y$

$= 3x^2y\ (x^2 - 6x + 9)$

$= 3x^2y\ (x-3)^2$

The factoring of the blue trinomial (2nd line) into the square of a binomial is the result of recognizing the special form it represents. Special forms are shown on the next two pages.

Factor: $6x^3y^3 - 24xy^3$

$= 6xy^3\ (x^2 - 4)$

$= 6xy^3\ (x+2)(x-2)$

The factoring of the blue binomial (2nd line) into binomials of lower degree is the result of recognizing the special form it represents. Special forms are shown on the next two pages.

Algebra
Special Forms of Quadratic Functions

It is helpful to be able to recognize a couple special forms of quadratic functions. In particular, if you can recognize perfect squares and differences of squares, your work will become easier and more accurate.

Perfect Squares

Perfect squares are of the form:
$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$

Identification and Solution

The following steps allow the student to identify and solve a trinomial that is a perfect square:

Step 1: Notice the first term of the trinomial is a square. Take its square root.

Step 2: Notice the last term of the trinomial is a square. Take its square root.

Step 3: Multiply the results of the first 2 steps and double that product. If the result is the middle term of the trinomial, the expression is a perfect square.

Step 4: The binomial in the solution is the sum or difference of the square roots calculated in steps 1 and 2. The sign between the terms of the binomial is the sign of the middle term of the trinomial.

Example:

$$\underset{\sqrt{4x^2} = \pm 2x}{\underset{\uparrow}{4x^2}} - 12xy + \underset{\sqrt{9y^2} = \pm 3y}{\underset{\uparrow}{9y^2}}$$

Notice that the middle term is double the product of the two square roots ($2x$ and $3y$). This is a telltale sign that the expression is a perfect square.

Identify the trinomial as a perfect square:

- Take the square roots of the first and last terms. They are $2x$ and $3y$.
- Test the middle term. Multiply the roots from the previous step, then double the result: $(2x \cdot 3y) \cdot 2 = 12xy$. The result (with a "$-$" sign in front) is the middle term of the original trinomial. Therefore, the expression is a perfect square.

To express the trinomial as the square of a binomial:

- The square roots of the first and last terms ($2x$ and $3y$) make up the binomial we seek.
- We may choose the sign of the first term, so let's choose the "$+$" sign.
- Having chosen the "$+$" sign for the first term, the second term of the binomial takes the sign of the middle term of the original trinomial ("$-$"). Therefore, the result is:

$$4x^2 - 12xy + 9y^2 = (2x - 3y)^2$$

Differences of Squares

Differences of squares are of the form: $\boldsymbol{a^2 - b^2 = (a+b)\cdot(a-b)}$

These are much easier to recognize than the perfect squares because there is no middle term to consider. Notice why there is no middle term:

$$\boldsymbol{(a+b)\cdot(a-b) = a^2 \underbrace{+\,ab - ab}_{\text{these two terms cancel}} - b^2 = a^2 - b^2}$$

Identification

To see if an expression is a difference of squares, you must answer "yes" to four questions:

1. Are there only two terms?
2. Is there a "$-$" sign between the two terms?
3. Is the first term a square? If so, take its square root.
4. Is the second term a square? If so, take its square root.

The solution is the product of a) the sum of the square roots in questions 3 and 4, and b) the difference of the square roots in steps 3 and 4.

Note: *A telltale sign of when an expression might be the difference of 2 squares is when the coefficients on the variables are squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, etc.*

Examples:

(1) $\boldsymbol{4x^2 - 25y^2 = (2x+5y)\cdot(2x-5y)}$

(2) $\boldsymbol{x^2 - 49 = (x+7)\cdot(x-7)}$

(3) $\boldsymbol{81 - 9z^2 = (9+3z)\cdot(9-3z)}$

(4) $\boldsymbol{\frac{x^2}{9} - \frac{y^2}{16} = \left(\frac{x}{3}+\frac{y}{4}\right)\cdot\left(\frac{x}{3}-\frac{y}{4}\right)}$

ADVANCED: Over the field of complex numbers, it is also possible to factor the sum of 2 squares:

$$\boldsymbol{a^2 + b^2 = (a+bi)\cdot(a-bi)}$$

This is not possible over the field of real numbers.

Algebra

Factoring Trinomials – Simple Case Method

A common problem in Elementary Algebra is the factoring of a trinomial that is neither a perfect square nor a difference of squares.

Consider the simple case where the coefficient of x^2 is 1. The general form for this case is:

$$(x+p)\cdot(x+q) = x^2 \underset{\substack{\uparrow \\ \text{sign 1}}}{+} \underbrace{(p+q)}_{\substack{\uparrow \\ \text{coefficient} \\ \text{of x}}}x \underset{\substack{\uparrow \\ \text{sign 2}}}{+} \underbrace{(pq)}_{\substack{\nwarrow \\ \text{constant}}}$$

In order to simplify the illustration of factoring a polynomial where the coefficient of x^2 is 1, we will use the orange descriptors above for the components of the trinomial being factored.

Simple Case Method

Step 1: Set up parentheses for a pair of binomials. Put "x" in the left hand position of each binomial.

Step 2: Put **sign 1** in the middle position in the left binomial.

Step 3: Multiply **sign 1** and **sign 2** to get the sign for the right binomial. Remember:

$$(+)\cdot(+)=(+) \qquad (-)\cdot(-)=(+)$$
$$(+)\cdot(-)=(-) \qquad (-)\cdot(+)=(-)$$

Step 4: Find two numbers that:

(a) Multiply to get the **constant**, and

(b) Add to get the **coefficient of x**

Fill in:

___ · ___ = ___

___ + ___ = ___

Step 5: Place the numbers in the binomials so that their signs match the signs from Steps 2 and 3. **This is the final answer.**

Step 6: Check your work by multiplying the two binomials to see if you get the original trinomial.

Example: Factor $x^2-3x-28$

$$=(x\qquad)\cdot(x\qquad)$$

$$=(x-\qquad)\cdot(x\qquad)$$

$$=(x-\qquad)\cdot(x+\qquad)$$

The numbers we seek are
4 and -7 because:
$4\cdot(-7)=-28$, and
$4-7=-3$

$$=(x-7)\cdot(x+4)$$

$$(x-7)\cdot(x+4)$$
$$=x^2+4x-7x-28$$
$$=x^2-3x-28$$

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Algebra
Factoring Trinomials – AC Method

There are times when the simple method of factoring a trinomial is not sufficient. Primarily this occurs **when the coefficient of x^2 is not 1**. In this case, you may use the AC method presented here, or you may use either the brute force method or the quadratic formula method (described on the next couple of pages).

AC Method

The AC Method derives its name from the first step of the process, which is to multiply the values of "a" and "c" from the general form of the quadratic equation: $\boldsymbol{y = ax^2 + bx + c}$

Step 1: Multiply the values of "a" and "c".

Step 2: Find two numbers that:

- (a) Multiply to get the **value of ac**, and
- (b) Add to get the **coefficient of x**

Fill in:

__ · __ = __

__ + __ = __

Step 3: Split the middle term into two terms, with coefficients equal to the values found in Step 2. (Tip: if only one of the coefficients is negative, put that term first.)

Step 4: Group the terms into pairs.

Step 5: Factor each pair of terms.

Step 6: Use the distributive property to combine the multipliers of the common term. **This is the final answer.**

Step 7: Check your work by multiplying the two binomials to see if you get the original trinomial.

Example: Factor $6x^2 - x - 2$

$$6x^2 - x - 2$$

$$-12$$

$$(-4) \cdot 3 = -12$$

$$(-4) + 3 = -1$$

$$6x^2 - 4x + 3x - 2$$

$$(6x^2 - 4x) + (3x - 2)$$

$$2x(3x - 2) + 1(3x - 2)$$

$$= \boldsymbol{(2x + 1) \cdot (3x - 2)}$$

$$(2x + 1) \cdot (3x - 2)$$
$$= 6x^2 - 4x + 3x - 2$$
$$= 6x^2 - x - 2$$

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Algebra
Factoring Trinomials – Brute Force Method

When the coefficient of x^2 is not 1, the factoring process becomes more difficult. There are a number of methods that can be used in this case.

If the question being asked is to find roots of the equation, and not to factor it, the student may want to use the quadratic formula whenever the coefficient of x^2 is not 1. Even if you are required to factor, and not just find roots, the quadratic formula may be a viable approach.

Brute Force Method

This method is exactly what it sounds like. Multiple equations are possible and you must try each of them until you find the one that works. Here are the steps to finding which equations are candidate solutions:

Step 1: Find all sets of whole numbers that multiply to get the coefficient of the first term in the trinomial. If the first term is positive, you need only consider positive factors.

Step 2: Find all sets of whole numbers that multiply to get the coefficient of the last term in the trinomial. You must consider both positive and negative factors.

Step 3: Create all possible products of binomials that contain the whole numbers found in the first two steps.

Step 4: Multiply the binomial pairs until you find one that results in the trinomial you are trying to factor.

Step 5: Identify the correct solution.

Example: Factor $\mathbf{4x^2+4x-3}$

Combinations that produce a product of 4 are:

$$1 \text{ and } 4 \quad \text{or} \quad 2 \text{ and } 2$$

Combinations that produce a product of -3 are:

$$-1 \text{ and } 3 \quad \text{or} \quad 1 \text{ and } -3$$

$$(x-1)(4x+3)$$
$$(x+1)(4x-3)$$
$$(x+3)(4x-1)$$
$$(x-3)(4x+1)$$
$$(2x-1)(2x+3)$$
$$(2x+1)(2x-3)$$

$$(x-1)(4x+3)=4x^2-x-3$$
$$(x+1)(4x-3)=4x^2+x-3$$
$$(x+3)(4x-1)=4x^2+11x-3$$
$$(x-3)(4x+1)=4x^2-11x-3$$
$$(2x-1)(2x+3)=4x^2+4x-3$$
$$(2x+1)(2x-3)=4x^2-4x-3$$

$$\mathbf{(2x-1)(2x+3)}=4x^2+4x-3$$

Notice the patterns in the candidate solutions in Step 4. Each pair of equations is identical except for the sign of the middle term in the product. Therefore, you can cut your work in half by considering only one of each pair until you see a middle term coefficient that has the right absolute value. If you have everything right but the sign of the middle term, switch the signs in the binomials to obtain the correct solution. **Remember to check your work!**

Algebra
Factoring Trinomials – Quadratic Formula Method

Quadratic Formula Method

The Quadratic Formula is designed specifically to find roots of a second degree equation. However, it can also be used as a back-door method to factor equations of second degree. The steps are:

Step 1: Apply the quadratic formula to determine the roots of the equation.

Step 2: Put each root into the form: $(x - root) = 0$.

Step 3: Show the two $(x - root)$ binomials as a product. Note that these binomials may contain fractions. We will eliminate the fractions, if possible, in the next step.

Step 4: Multiply the binomials in Step 3 by the coefficient of x^2 the following way:

- (a) Break the coefficient of x^2 into its prime factors.
- (b) Allocate the prime factors to the binomials in a way that eliminates the fractions.

Step 5: Check your work.

Example:

Factor: $\mathbf{4x^2 + 4x - 3}$

Step 1: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-4\pm\sqrt{4^2-4(4)(-3)}}{2(4)} = \frac{-4\pm\sqrt{64}}{8} = \frac{-4\pm 8}{8} = -\frac{3}{2}$ or $\frac{1}{2}$

Step 2: The two equations containing roots are: $\left(x + \frac{3}{2}\right) = 0$ and $\left(x - \frac{1}{2}\right) = 0$.

Step 3: $\left(x + \frac{3}{2}\right)\left(x - \frac{1}{2}\right)$

Step 4: The coefficient of x^2 in the original equation is 4, and $4 = 2 \cdot 2$. An inspection of the binomials in Step 3 indicates we need to multiply each binomial by 2 in order to eliminate the fractions:

$$2 \cdot \left(x + \frac{3}{2}\right) = (2x + 3) \qquad \text{and} \qquad 2 \cdot \left(x - \frac{1}{2}\right) = (2x - 1)$$

So that: $4x^2 + 4x - 3 = \mathbf{(2x + 3) \cdot (2x - 1)}$ in factored form

Step 5: Check (using FOIL) $(2x + 3) \cdot (2x - 1) = 4x^2 - 2x + 6x - 3 = 4x^2 + 4x - 3$, which is the equation we were trying to factor.

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Algebra
Solving Equations by Factoring

There are a number of reasons to factor a polynomial in algebra; one of the most common reasons is to find the zeroes of the polynomial. A “zero” is a domain value (e.g., x-value) for which the polynomial generates a value of zero. Each zero is a solution of the polynomial.

In factored form, it is much easier to find a polynomial’s zeroes. Consider the following:

$(x-2)(x+4)(x-8)(x-\pi)(x+3)$ is the factored form of a polynomial.

If a number of items are multiplied together, the result is zero whenever any of the individual items is zero. This is true for constants and for polynomials. Therefore, if any of the factors of the polynomial has a value of zero, then the whole polynomial must be zero. We use this fact to find zeroes of polynomials in factored form.

Example 1:

Find the zeroes of $y=(x-2)(x+4)(x-8)(x-\pi)(x+3)$.

Step 1: Set the equation equal to zero.

$$(x-2)(x+4)(x-8)(x-\pi)(x+3)=0$$

Step 2: The whole equation is zero whenever any of its factors is zero. For the example, this occurs when:

$$\left.\begin{aligned}(x-2)&=0, \text{ or}\\(x+4)&=0, \text{ or}\\(x-8)&=0, \text{ or}\\(x-\pi)&=0, \text{ or}\\(x+3)&=0\end{aligned}\right\}$$

The solution set, then, is:

$$x=\{2,-4,8,\pi,-3\}$$

or, more conventionally, the x-values are put in numerical order from smallest to largest:

$$x=\{\,-4,-3,2,\pi,8\}$$

Set Notation: We may list the set of solutions to a problem by placing the solutions in braces {}, separated by commas.

Example 2:

Find the zeroes of $y=x^2-7x+6$

$$x^2-7x+6=0$$

$$(x-6)(x-1)=0$$

$$(x-6)=0 \qquad\qquad (x-1)=0$$

$$x=6 \qquad\qquad x=1$$

The solution set contains the two domain values that make the original equation zero, namely:

$$x=\{1,6\}$$